Proof by Contradiction: A Classical Example

What we will prove. Let us prove that the square root of 2 is not a rational number.

Why this is important. The proof comes from the school of Pythagoras.

For anyone who studied at school, Pythagoras is mostly known for his theorem: that the square of the hypothenuse is equal to the sum of the squares of the sides: $c^2 = a^2 + b^2$. In particular, if we take a right triangle with sides equal to 1, then the length of its hypothenuse – i.e., of the diagonal of the unit square – is equal to $\sqrt{2}$.

Those who studied music also know that Pythagoras discovered that the sounds produced by two strings sound nice together nice if and only if the ratio of their lengths is a rational number, i.e., a ratio of two integers (especially the ratio of two small integers). This and other discoveries led Pythagoras to believe that everything in the world is described by rational numbers.

Because of this belief, for several years, he tried to prove that $\sqrt{2}$ – the length of the diagonal of the unit square – is a rational number. He could not prove it, his students could not prove it – until one of his students proved that $\sqrt{2}$ is *not* a rational number.

This was such a shock to Pythagoras – since it contradicted his fundamental beliefs – that for many years, he and his students kept this result a secret. A legend says that they even killed a colleague who wanted to disclose this secret. The secret became known only after Pythagoras died.

Proof. Let us prove it by contradiction. Let us assume that $\sqrt{2}$ is a rational number, i.e., $\sqrt{2} = m/n$ for some integers m and n.

If the numbers m and n have a common factor, then we can divide both m and n by this factor and get the values a and b for which $m/n = a/b$ and for which a and b have no common factors. For these values a and b, we have $\sqrt{2} = a/b$.

Let us now get a contradiction.

- Multiplying both sides of the above equality by b, we get $\sqrt{2} \cdot b = a$.
- Squaring both sides, we get $2b^2 = a^2$.
- The left-hand side $2b^2$ of this equality is divisible by 2. Since $a^2 = 2b^2$, the value a^2 is the same number as $2b^2$, so the value $a^2 = a \cdot a$ must also be divisible by 2.

- Thus, a is divisible by 2, i.e. the ratio $a/2$ is an integer. Let us denote this integer by p. From $a/2 = p$, we can conclude that $a = 2p$.
- For $a = 2p$, we have

$$a^2 = (2p) \cdot (2p) = (2 \cdot p) \cdot (2 \cdot p) = (2 \cdot 2) \cdot (p \cdot p) = 4p^2.$$

- Substituting $a^2 = 4p^2$ into the formula $2b^2 = a^2$, we get $2b^2 = 4p^2$.
- Dividing both sides by 2, we get $b^2 = 2p^2$.
- The right-hand side of this equality is divisible by 2, so the left-hand side $b^2 = b \cdot b$ must also be divisible by 2.
- Thus, b is divisible by 2.
- So, a and b have a common factor 2 – which contradicts to the fact that a and b have no common factors.

This contradiction shows that our original assumption – that $\sqrt{2}$ is a rational number – is wrong. The statement is proven.

Important comment. In this proof, we used the fact that:

- if a product of two or more numbers is divisible by 2,
- then one of them should be divisible by 2.

This property is true for any prime number. For example:

- if the product of two or more numbers is divisible by 3,
- then one of them is divisible by 3

However, this property is *not* true for numbers which are not prime, such as 6. For example:

- the product $2 \cdot 3$ of two numbers 2 and 3 is divisible by 6
- but neither of the numbers 2 or 3 is divisible by 6.

So, e.g., to prove that $\sqrt{6}$ is not a rational number, we need to consider:

- either divisibility by 2
- or divisibility by 3.

In general, we need to consider divisibility with respect to some prime factor of the corresponding number.