Solution to Homework Problem 23

Homework problem 23. Prove that the cubic root of 5 is not a rational number.

Solution. Let us prove it by contradiction. Let us assume that $\sqrt[3]{5}$ is a rational number, i.e., $\sqrt[3]{5} = a/b$ for some integers a and b.

If the numbers a and b have a common factor, then we can divide both a and b by this factor and get the same ratio. Thus, we can always find a and b that have no common factors.

Let us now get a contradiction.

- Multiplying both sides of the above equality by b, we get $\sqrt[3]{5} \cdot b = a$.
- Cubic both sides, we get $5b^3 = a^3$.
- The left-hand side of this equality is divisible by 5, so the right-hand side $a^3 = a \cdot a \cdot a$ must also be divisible by 5.
- Thus, a is divisible by 5, i.e., $a = 5p$ for some integer p.
- For $a = 5p$, we have $a^3 = (5p) \cdot (5p) \cdot (5p) = 5^3 \cdot p^3$.
- Substituting $a^3 = 5^3 \cdot p^3$ into the formula $5b^3 = a^3$, we get $5b^3 = 5^3 \cdot p^3$.
- Dividing both sides by 5, we get $b^3 = 5^2 \cdot p^3$.
- The right-hand side of this equality is divisible by 5, so the left-hand side $b^3 = b \cdot b \cdot b$ must also be divisible by 5.
- Thus, b is divisible by 5.
- So, a and b have a common factor 5 – which contradicts to the fact that a and b have no common factors.

This contradiction shows that our original assumption – that $\sqrt[3]{5}$ is a rational number – is wrong. The statement is proven.