Solution to Homework Problem 23

Homework problem 23. Prove that the cubic root of 5 is not a rational number.

Solution. Let us prove it by contradiction. Let us assume that \( \sqrt[3]{5} \) is a rational number, i.e., \( \sqrt[3]{5} = a/b \) for some integers \( a \) and \( b \).

If the numbers \( a \) and \( b \) have a common factor, then we can divide both \( a \) and \( b \) by this factor and get the same ratio. Thus, we can always find \( a \) and \( b \) that have no common factors.

Let us now get a contradiction.

- Multiplying both sides of the above equality by \( b \), we get \( \sqrt[3]{5} \cdot b = a \).
- Cubing both sides, we get \( 5b^3 = a^3 \).
- The left-hand side of this equality is divisible by 5, so the right-hand side \( a^3 = a \cdot a \cdot a \) must also be divisible by 5.
- Thus, \( a \) is divisible by 5, i.e., \( a = 5p \) for some integer \( p \).
- For \( a = 5p \), we have \( a^3 = (5p) \cdot (5p) \cdot (5p) = 5^3 \cdot p^3 \).
- Substituting \( a^3 = 5^3 \cdot p^3 \) into the formula \( 5b^3 = a^3 \), we get \( 5b^3 = 5^3 \cdot p^3 \).
- Dividing both sides by 5, we get \( b^3 = 5^2 \cdot p^3 \).
- The right-hand side of this equality is divisible by 5, so the left-hand side \( b^3 = b \cdot b \cdot b \) must also be divisible by 5.
- Thus, \( b \) is divisible by 5.
- So, \( a \) and \( b \) have a common factor 5 – which contradicts to the fact that \( a \) and \( b \) have no common factors.

This contradiction shows that our original assumption – that \( \sqrt[3]{5} \) is a rational number – is wrong. The statement is proven.