Solution to Homework 11

Task. In Homework 6, we considered the following pushdown automaton. This pushdown automaton has four states:

- the starting state \( s \),
- the state \( a \) meaning that the number of \( A \)s is larger than or equal to the number of \( B \)s,
- the state \( b \) meaning that the number of \( B \)s is larger than or equal to the number of \( A \)s, and
- the final state \( f \).

The transitions are as follows:

- From \( s \) to \( a \), the transition is \( \varepsilon, \varepsilon \rightarrow \varepsilon \).
- From \( a \) to \( a \), the transitions are: \( A, \varepsilon \rightarrow A \) and \( A, B \rightarrow \varepsilon \).
- From \( a \) to \( b \), the transition is \( B, \varepsilon \rightarrow B \).
- From \( b \) to \( b \), the transitions are: \( B, \varepsilon \rightarrow B \) and \( B, A \rightarrow \varepsilon \).
- From \( b \) to \( a \), the transition is \( A, \varepsilon \rightarrow A \).
- From \( a \) to \( f \), the transition is \( \varepsilon, \varepsilon \rightarrow \varepsilon \).
- From \( b \) to \( f \), the transition is \( \varepsilon, \varepsilon \rightarrow \varepsilon \).

Use the general algorithm to transform this pushdown automaton into a context-free grammar. Show, step-by-step, how the resulting grammar will generate the word \( ABAB \).

Solution. Let us recall how the word \( ABAB \) is accepted by this automaton. We will list consequent states and the contents of the corresponding stacks, described from the top to bottom, and what symbols we see in the corresponding transitions:

- state \( s \), stack is empty;
we jump from state \( s \) to state \( a \);
- state \( a \), stack is empty;
we read $A$;
  - state $a$, stack has $A$;
we read $B$;
  - state $a$, stack is empty;
we read $A$;
  - state $a$, stack has $A$;
we read $B$;
  - state $a$, stack is empty;
we jump from state $a$ to state $f$;
  - state $f$, stack is empty.

These transitions can be described as follows:

<table>
<thead>
<tr>
<th>state</th>
<th>$s$</th>
<th>$a$</th>
<th>$a$</th>
<th>$a$</th>
<th>$a$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>read</td>
<td></td>
<td>$A$</td>
<td>$B$</td>
<td>$A$</td>
<td>$B$</td>
<td></td>
</tr>
<tr>
<td>stack</td>
<td></td>
<td>$A$</td>
<td></td>
<td></td>
<td></td>
<td>$A$</td>
</tr>
</tbody>
</table>

We start with the state $s$, we end up in the final state $f$. Thus, the first rule we apply if the rule $S \rightarrow A_{sf}$:

$$
\begin{align*}
S \\
\quad A_{sf}
\end{align*}
$$

Here, we have three intermediate situations with empty stacks. In general, when a transition form $p$ to $r$ goes through the state $q$ with an empty stack, we need to use the transitivity rule $A_{pq} \rightarrow A_{pq}A_{qr}$. In our case, the first intermediate state with an empty stack is the state $a$, so we use the rule $A_{sf} \rightarrow A_{sa}A_{af}$, so the derivation so far takes the following form:

$$
\begin{align*}
S \\
\quad A_{sf} \\
\quad \quad A_{sa} \quad A_{af}
\end{align*}
$$
Transition from $s$ to $a$ follows the jump rule. We do not push anything, so there is no similar rule for popping. As recommended in this case, we pair this rule with the trivial transition from $a$ to $a$, in which we do not push anything and do not pop anything:

$\begin{align*}
\text{Transition from } s \text{ to } a \\
\text{Pair with trivial transition from } a \text{ to } a
\end{align*}$

In general, we have the two transitions

$\begin{align*}
\text{Pair with general transitions for } p, q, r, x, y, z
\end{align*}$

What do we need to plug in instead of $p, q, r, x, y, z$ in the general 2-rule picture to come up with this particular picture:

- instead of $p$, we place $s$;
- instead of $q, r$, and $s$ we place $a$;
- instead of $x, y, t$, we place $\epsilon$.

If we make these substitutions in the general rule:

$A_{ps} \rightarrow xA_{qr}y,$

we get the rule

$A_{sa} \rightarrow \epsilon A_{aa} \epsilon.$

Since concatenation with the empty string does not change anything, this means

$A_{sa} \rightarrow A_{aa}.$

Thus, the derivation so far takes the following form:

$S \rightarrow A_{sf} \rightarrow A_{sa} A_{af} \rightarrow A_{aa} A_{aa} \rightarrow A_{aa}$
Here, $A_{aa}$ means immediate transition from the state $a$ to itself, so we can use the rule $A_{aa} \rightarrow \varepsilon$:

We covered the transition from $s$ to $a$:

<table>
<thead>
<tr>
<th>state</th>
<th>s</th>
<th>a</th>
<th>a</th>
<th>a</th>
<th>a</th>
<th>a</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>read</td>
<td></td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>stack</td>
<td></td>
<td>A</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now, we need to cover the remaining transition from $a$ to $f$. In this transition, we go through yet another state with an empty stack, so we need to use the transitivity rule $A_{af} \rightarrow A_{aa}A_{af}$:
In the first transition, we first push $A$, and then immediately pop this $A$. Thus, we have the following combination of push-pop rules:

$$
\begin{array}{c}
a \xrightarrow{A, \varepsilon \rightarrow A} a \\
a \xrightarrow{B, A \rightarrow \varepsilon} a
\end{array}
$$

In general, we have the two transitions

$$
\begin{array}{c}
p \xrightarrow{x, \varepsilon \rightarrow t} q \\
r \xrightarrow{y, t \rightarrow \varepsilon} s
\end{array}
$$

What do we need to plug in instead of $p, q, r, s$ in the general 2-rule picture to come up with this particular picture:

- instead of $p, q, r$, and $s$, we place $a$;
- instead of $x$ and $t$, we place $A$;
- instead of $y$, we place $B$.

If we make these substitutions in the general rule:

$$A_{ps} \rightarrow x_{A_{q}y},$$

we get the rule

$$A_{aa} \rightarrow AA_{aa}B.$$

Thus, the derivation so far takes the following form:

Here, the remaining transition between $a$ and $a$ does not including any additional steps, so we can use the rule $A_{aa} \rightarrow \varepsilon$: 

$$5$$
We covered the transition from $s$ to $a$:

<table>
<thead>
<tr>
<th>state</th>
<th>$s$</th>
<th>$a$</th>
<th>$g$</th>
<th>$a$</th>
<th>$a$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>read</td>
<td>$A$</td>
<td>$B$</td>
<td>$A$</td>
<td>$B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>stack</td>
<td>$A$</td>
<td>$A$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now, we need to cover the remaining transition from $a$ to $f$. In this transition, we also have an intermediate state $a$ with an empty stack, so we need to use transitivity $A_{af} \rightarrow A_{aa}A_{af}$:
For the transition from $a$ with an empty stack to $a$ with an empty stack, we get the same rule $A_{aa} \rightarrow AA_{aa}B$ as before. For the jump from $a$ to $f$, we get a transition similar to the jump from $s$ to $a$: $A_{af} \rightarrow A_{ff} \rightarrow \varepsilon$. Thus, we arrive at the following derivation:

```
S

A_{af}

A_{aa}      A_{aa}

A_{aa}      A_{aa}    A_{aa}

\varepsilon   A   B  A_{aa}    A_{af}

\varepsilon   A   B  A_{aa}    A_{af}

\varepsilon     B     A_{aa}   A_{ff}

\varepsilon     B     A_{aa}   A_{ff}

\varepsilon
```