Solution to Homework 11

Task. In Homework 6, we considered the following pushdown automaton. This pushdown automaton has four states:

- the starting state s,
- the state a meaning that the number of As is larger than or equal to the number of Bs,
- the state b meaning that the number of Bs is larger than or equal to the number of As, and
- the final state f.

The transitions are as follows:

- From s to a, the transition is $\varepsilon, \varepsilon \to \varepsilon$.
- From a to a, the transitions are: $A, \varepsilon \to A$ and $A, B \to \varepsilon$.
- From a to b, the transition is $B, \varepsilon \to B$.
- From b to b, the transitions are: $B, \varepsilon \to B$ and $B, A \to \varepsilon$.
- From b to a, the transition is $A, \varepsilon \to A$.
- From a to f, the transition is $\varepsilon, \varepsilon \to \varepsilon$.
- From b to f, the transition is $\varepsilon, \varepsilon \to \varepsilon$.

Use the general algorithm to transform this pushdown automaton into a context-free grammar. Show, step-by-step, how the resulting grammar will generate the word $ABAB$.

Solution. Let us recall how the word $ABAB$ is accepted by this automaton. We will list consequent states and the contents of the corresponding stacks, described from the top to bottom, and what symbols we see in the corresponding transitions:

- state s, stack is empty;

we jump from state s to state a;

- state a, stack is empty;

we read A;

- state a, stack has A;

we read B;

- state a, stack is empty;

we read A;

- state a, stack has A;

we read B;

- state a, stack is empty;

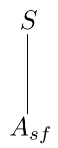
we jump from state a to state f;

- state f, stack is empty.

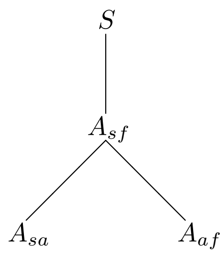
These transitions can be described as follows:

state	s	a	a	a	a	a	f
read			A	B	A	B	
stack			A		A		

We start with the state s, we end up in the final state f. Thus, the first rule we apply if the rule $S \to A_{sf}$;



Here, we have three intermediate situations with empty stacks. In general, when a transition form p to r goes through the state q with an empty stack, we need to use the transitivity rule $A_{pq} \to A_{pq}A_{qr}$. In our case, the first intermediate state with an empty stack is the state a, so we use the rule $A_{sf} \to A_{sa}A_{af}$, so the derivation so far takes the following form:



Transition from s to a follows the jump rule. We do not push anything, so there is no similar rule for popping. As recommended in this case, we pair this rule with the trivial transition from a to a, in which we do not push anything and do not pop anything:

In general, we have the two transitions

What do we need to plug in instead of p, q, etc. in the general 2-rule picture to come up with this particular picture:

- instead of p, we place s;
- instead of q, r, and s we place a;
- instead of x, y, and t, we place ε.

If we make these substitutions in the general rule:

$$A_{ps} \to xA_{qr}y,$$

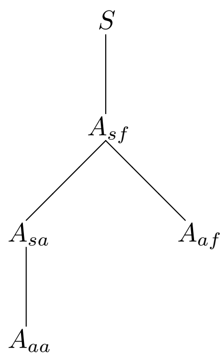
we get the rule

$$A_{sa} \to \varepsilon A_{aa}\varepsilon.$$

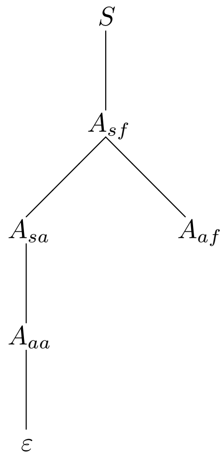
Since concatenation with the empty string does not change anything, this means

$$A_{sa} \to A_{aa}.$$

Thus, the derivation so far takes the following form:



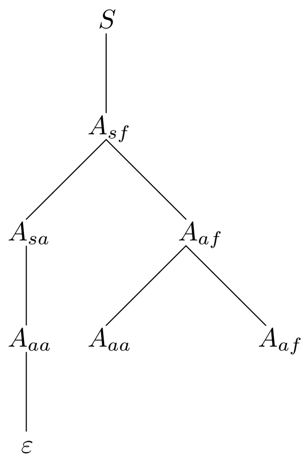
Here, A_{aa} means immediate transition from the state a to itself, so we can use the rule $A_{aa} \to \varepsilon$:



We covered the transition from s to a:

state	$\underline{s}$	$\underline{a}$	a	a	a	a	f
read			A	B	A	B	
stack			A		A		

Now, we need to cover the remaining transition from a to f. In this transition, we go though yet another state with an empty stack, so we need to use the transitivity rule $A_{af} \to A_{aa}A_{af}$:



In the first transition, we first push A, and then immediately pop this A. Thus, we have the following combination of push-pop rules:

In general, we have the two transitions

What do we need to plug in instead of p, q, etc. in the general 2-rule picture to come up with this particular picture:

- instead of p, q, r, and s, we place a;
- instead of x and t, we place A;
- instead of y, we place B.

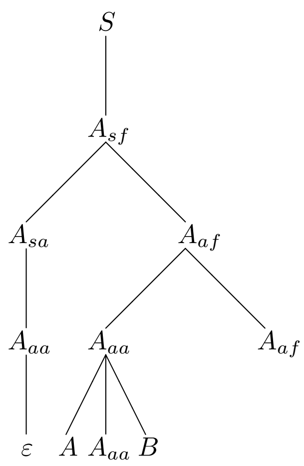
If we make these substitutions in the general rule:

$$A_{ps} \to xA_{qr}y,$$

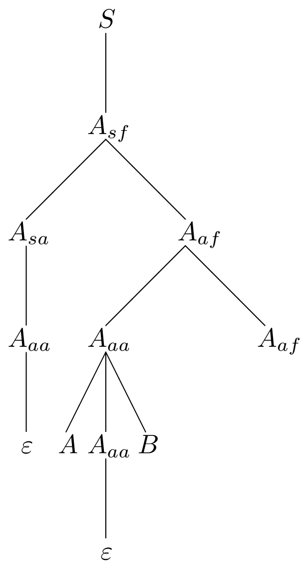
we get the rule

$$A_{aa} \to AA_{aa}B.$$

Thus, the derivation so far takes the following form:



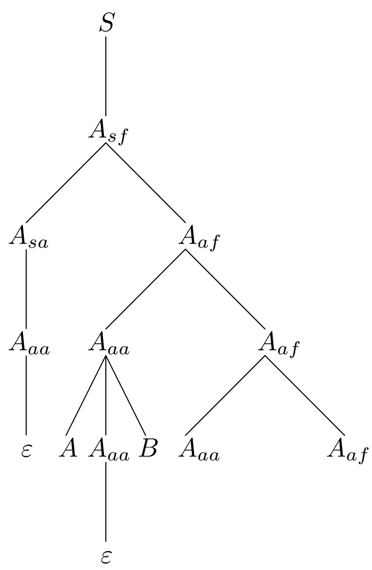
Here, the remaining transition between a and a does not include any additional steps, so we can use the rule $A_{aa} \to \varepsilon$:



We covered the transition from s to a:

state	$\underline{s}$	$\underline{a}$	$\underline{a}$	$\underline{a}$	a	a	f
read			A	B	A	B	
stack			A		A		

Now, we need to cover the remaining transition from a to f. In this transition, we also have an intermediate state a with an empty stack, so we need to use transitivity $A_{af} \to A_{aa}A_{af}$:



For the transition from a with an empty stack to a with an empty stack, we get the same rule $A_{aa} \to AA_{aa}B$ as before. For the jump from a to f, we get a transition similar to the jump from s to a: $A_{af} \to A_{ff} \to \varepsilon$. Thus, we arrive at the following derivation: