Solution to Homework 4

Task: A balanced budget means:

- that you only spend money that you earn and
- that at the end, you will spend all the money you earned.

We can describe each budget by a sequence of symbols s (meaning spending a dollar) and e (meaning earning a dollar). For example:

- the sequence *eeesesss* is balanced but
- the sequence *esseseee* is not, since, according to this sequence, after earning 1 dollar we intend to spend two dollars – and we only have one dollar earned so far.

Prove that the language of all the words that correspond to a balanced budget is not regular.¡p¿

Solution. We will prove this result by contradiction. Let us assume that the language L is regular, and let us show that this assumption leads to a contradiction.

Since this language is regular, according to the Pumping Lemma, there exists an integer p such that every word from L whose length $\text{len}(w)$ is at least p can be represented as a concatenation $w = xyz$, where:

- y is non-empty;
- the length $\text{len}(xy)$ does not exceed p, and
- for every natural number i, the word $xy^iz \stackrel{\text{def}}{=} xy\ldots yz$, in which y is repeated i times, also belongs to the language L.

Let us take the word

$$w = e^p s^p = e\ldots es\ldots s,$$

in which first the letter e is repeated p times and then the letter s is repeated p times. The length of this word is $p + p = 2p > p$. So, by pumping lemma, this word can be represented as $w = xyz$ with $\text{len}(xy) \le p$. The word $w = xyz$ starts with xy, and the length of xy is smaller than or equal to p. Thus, xy is among the first p symbols of the word w – and these symbols are all e's. So, the word y only has e's.

In the original word $w = xyz$, we had e repeated p times and s repeated p times. When we go from the word $w = xyz$ to the word $xyyz$, we add e's, and we do not add any s's. Thus, we still have s repeated p times, but the number of times e is repeated is now larger than p. In any word from the language L, we should have exactly as many s's as e's – otherwise, the sequence will not be balanced. So, in the word $xyyz$, this balance is disrupted. Thus, the word $xyyz$ cannot be in the language L.

On the other hand, by Pumping Lemma, the word $xyyz$ must be in the language L. So, we proved two opposite statements:

- that this word *is not* in L and
- that this word *is* in L.

This is a contradiction.

The only assumption that led to this contradiction is that L is a regular language. Thus, this assumption is false, so the language L is not regular.