Solution to Problem 6

**Task:** Show, step by step, how the following pushdown automaton – that checks whether a word consisting of letters $e$ and $s$ corresponds to a balanced budget – will accept the word $eses$. This pushdown automaton has three states:

- the starting state $s$,
- the state $b$ meaning that so far, we have earned at least as much as we spent, and
- the final state $f$.

The transitions are as follows:

- From $s$ to $b$, the transition is $\varepsilon, \varepsilon \rightarrow \$; 
- From $b$ to $b$, the transitions are: $e, \varepsilon \rightarrow d$ and $s, d \rightarrow \varepsilon$.
- From $b$ to $f$, the transition is: $\varepsilon, \$ \rightarrow \varepsilon$.

**Solution.** Our pushdown automaton has the following form:

\[
\begin{array}{c}
  s \\
  e, \varepsilon \rightarrow \$
\end{array} 
\hspace{1cm} 
\begin{array}{c}
  b \\
  e, \varepsilon \rightarrow d \\
  s, d \rightarrow \varepsilon \\
  \varepsilon, \$ \rightarrow \varepsilon
\end{array} 
\hspace{1cm} 
\begin{array}{c}
  f \\
  \varepsilon, \varepsilon \rightarrow \$
\end{array}
\]

We start in the starting state $s$ with an empty stack:

\[
\begin{array}{c}
  s \\
  e, \varepsilon \rightarrow \$
\end{array} 
\hspace{1cm} 
\begin{array}{c}
  b \\
  e, \varepsilon \rightarrow d \\
  s, d \rightarrow \varepsilon \\
  \varepsilon, \$ \rightarrow \varepsilon
\end{array} 
\hspace{1cm} 
\begin{array}{c}
  f \\
  \varepsilon, \varepsilon \rightarrow \$
\end{array}
\]

We want to eventually reach the final state, and the only way to get out of the starting state is to use the rule $\varepsilon, \varepsilon \rightarrow \$ and go to state $b$.
The stack now contains the dollar sign. After that, we read the first symbol $e$ and thus, use the rule $e, \varepsilon \rightarrow d$:

The stack now contains a symbol $d$ on top of the dollar sign. Then, we read the next symbol $s$ and use the rule $s, d \rightarrow \varepsilon$:

Because of this rule, the top symbol $d$ is popped out of the stack, so we only have the dollar sign in the stack. Now, we again read the symbol $e$ and use the rule $e, \varepsilon \rightarrow d$:

The stack now contains a symbol $d$ on top of the dollar sign. Then, we read the last symbol $s$ and use the rule $s, d \rightarrow \varepsilon$: 

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We have read all the symbols of the word. The stack now contains only one symbol – the dollar sign. So, to get to the final state, we can use the rule $\varepsilon, \$ \rightarrow \varepsilon$:

$$
\begin{align*}
\varepsilon, \varepsilon & \rightarrow d \\
\varepsilon, \$ & \rightarrow \varepsilon \\
\end{align*}
$$

We are now in the final state with an empty stack, so the word *eses* is accepted.

To illustrate these transitions, let us list all the symbols we read, all the states that this automaton goes through, and under each state, the contents of the corresponding stack, with $\rightarrow$ indicating transition corresponding to reading a symbol:

<table>
<thead>
<tr>
<th>read</th>
<th>e</th>
<th>s</th>
<th>e</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>state</td>
<td>s</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>stack</td>
<td>$$</td>
<td>$$</td>
<td>$$</td>
<td>$$</td>
</tr>
</tbody>
</table>

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