Solution to Homework 9

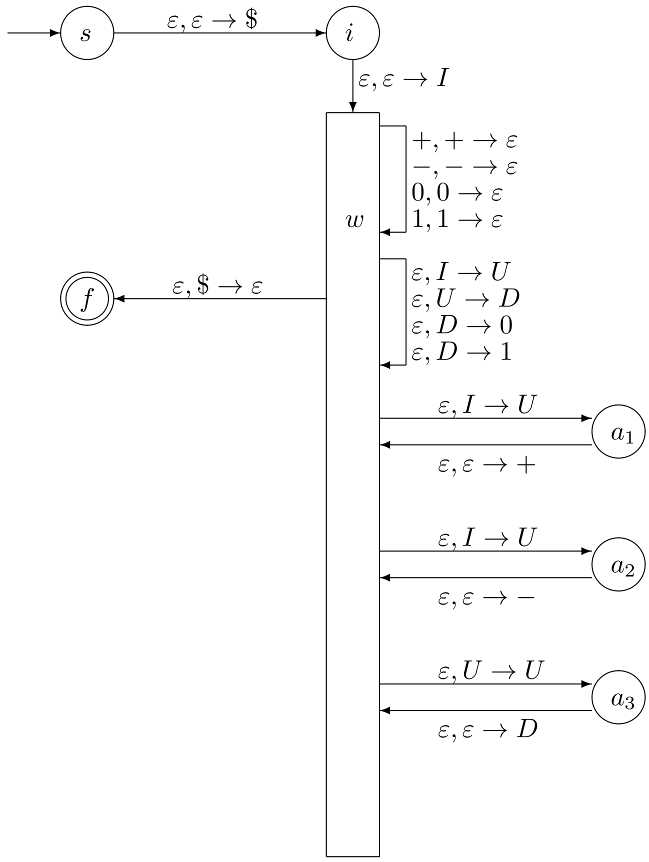
Background. In Problem 7, we considered a grammar with rules $I \to +U$, $I \to -U$, $I \to U$, $U \to DU$, $U \to D$, $D \to 0$, and $D \to 1$.

Tasks:

1. Use a general algorithm to construct a (non-deterministic) pushdown automaton that corresponds to context-free grammar described in Problem 7.

2. Show, step by step, how the word $+101$ will be accepted by this automaton.

Solution to Task 1. By using the general algorithm, we get the following pushdown automaton:



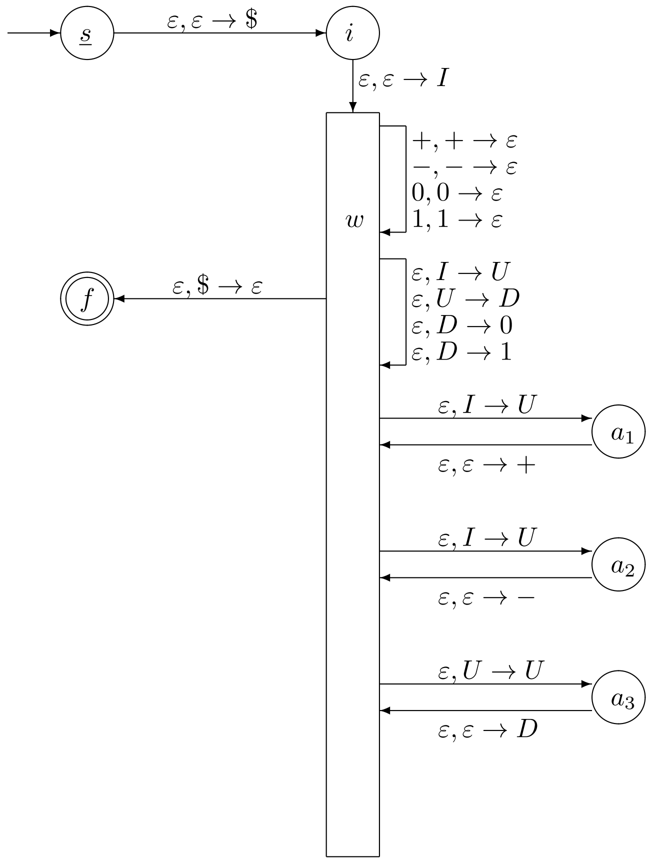
Solution to Task 2. Let us show how this is done on the example of the word +110 generated by the above automaton:

$$\underline{I} \to +\underline{U} \to +\underline{D}U \to +1\underline{U} \to +1\underline{D}U \to +11\underline{U} \to +11\underline{D} \to +110.$$

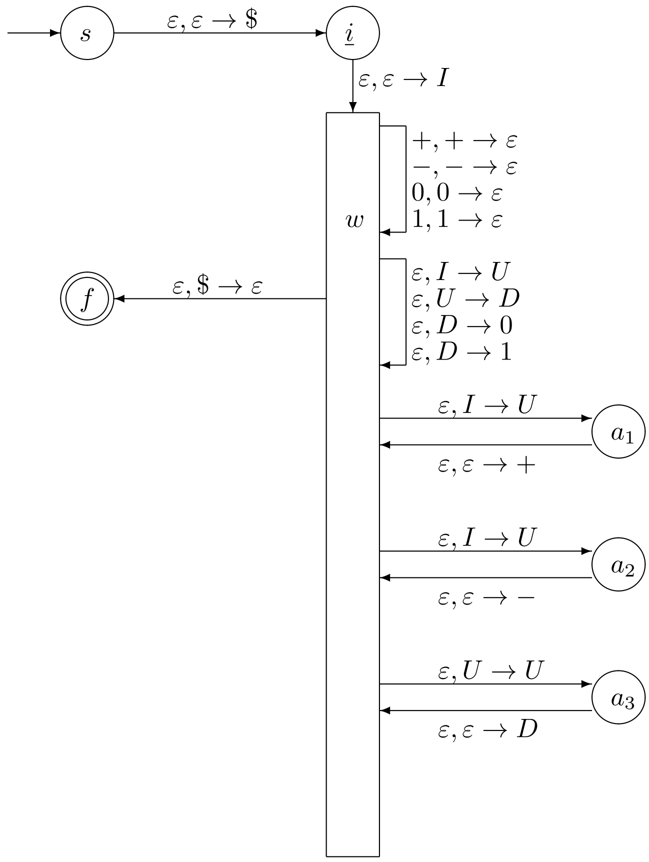
To make this derivation clearer, let us mark the variables U and D corresponding to different transitions by subscripts:

$$\underline{I} \to +\underline{U_1} \to +\underline{D_1}U_2 \to +1\underline{U_2} \to +1\underline{D_2}U_3 \to +11\underline{U_3} \to +11\underline{D_3} \to +110.$$

Let us now trace what our pushdown automaton will do. We start in the state s with an empty stack:



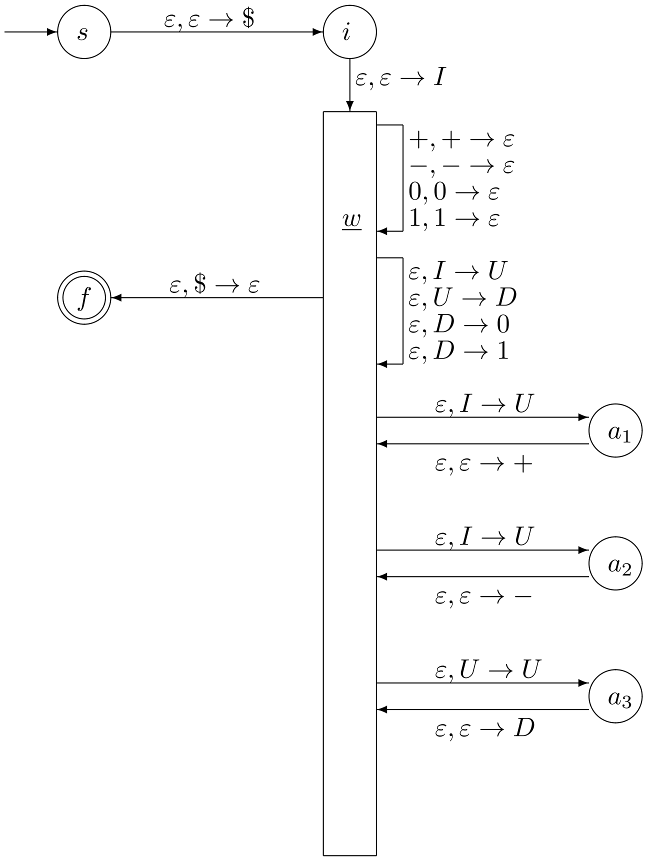
The only thing we can do when in the state s is push the dollar sign into the stack and get to the intermediate state i:



The contents of the stack is as follows:

$

When we are in the state i, the only thing we can do is push the starting variable I into the stack and go into the working state w;



Now, the stack contains the starting variable on top of the dollar sign:

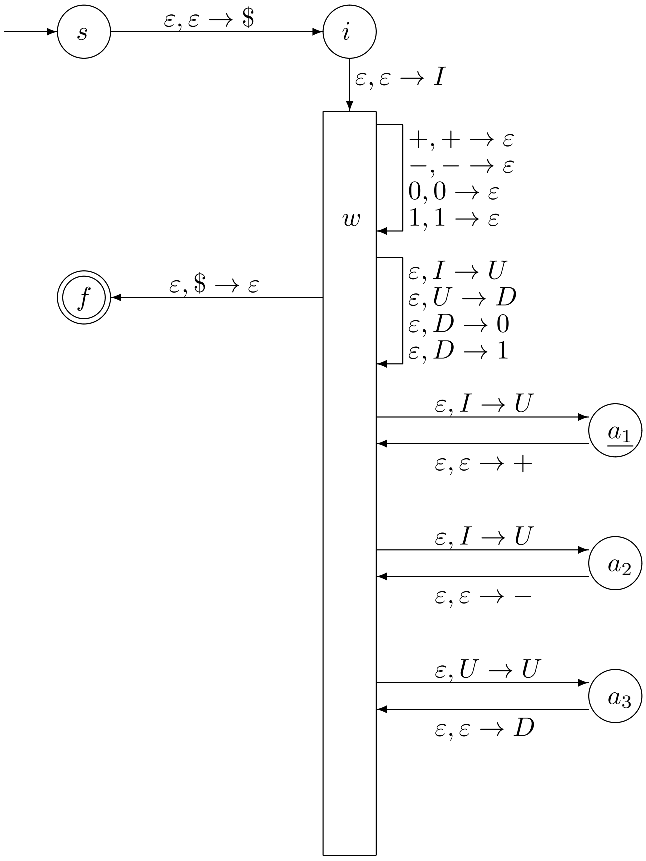
I
\$

Now that we are in the working state, we can start following the rules that were used to derive the word +110. The first rule was $I \to +U$, or, to be precise, $I \to +U_1$. As we have mentioned, this rule is implemented in two steps:

- first, we pop I and push the last symbol of the right-hand side – in this cases, the symbol U (that corresponds to the first occurrence U_1) – into the stack, getting into the auxiliary state a_1;
- then, we push $+$ into the stack, and go back to the working state w.

Let us illustrate this step by step.

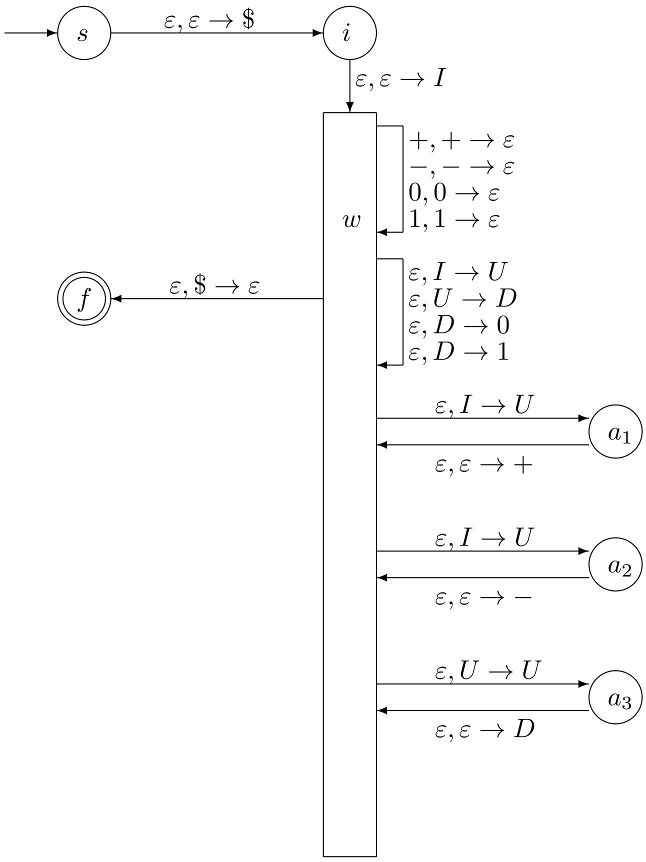
First, we pop I, push U, and go into the state a_1:



The stack will now have U instead of the original I:

U
$\$$

Then, we push $+$ into the stack and go back to working state w:



The stack will now have + on top of its previous contents:

$+$
U
$\$$

Now, the symbol + is top of the stack. The only thing we can do if a terminal symbol is on top of the stack is use one of the rules of the type $x, x \to \varepsilon$ where x stands for the corresponding terminal symbol.

In our case:

- since the terminal symbol on top of the stack is the symbol +,
- we need to use the rule $+, + \to \varepsilon$,

i.e., we read the symbol + from the original word +110 and pop the top symbol + from the stack. We remain in the same state w, but the stack changes. The stack now has the following form:

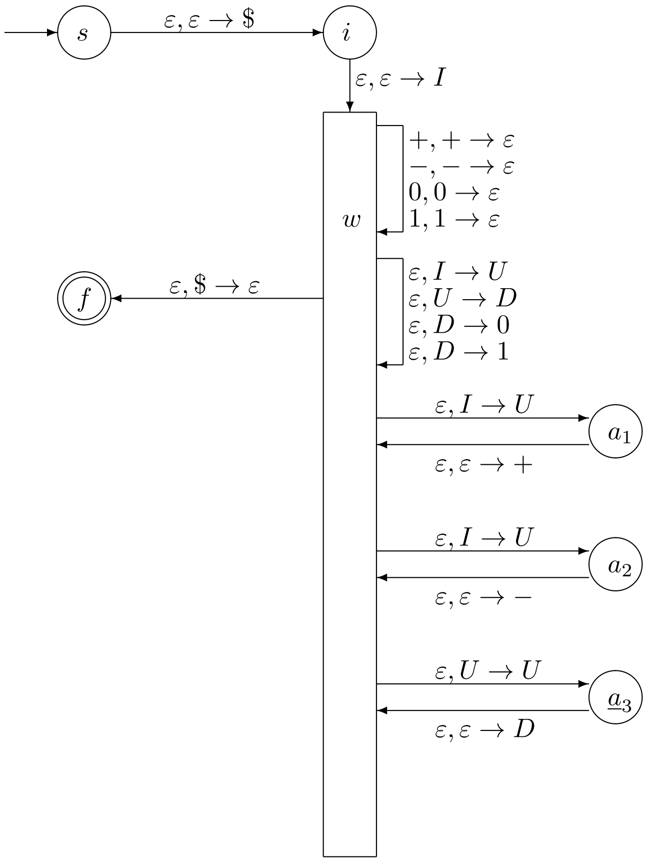
U
$\$$

In the derivation of our word, next, we use the rule $U \to DU$. As before, this rule is implemented in two steps:

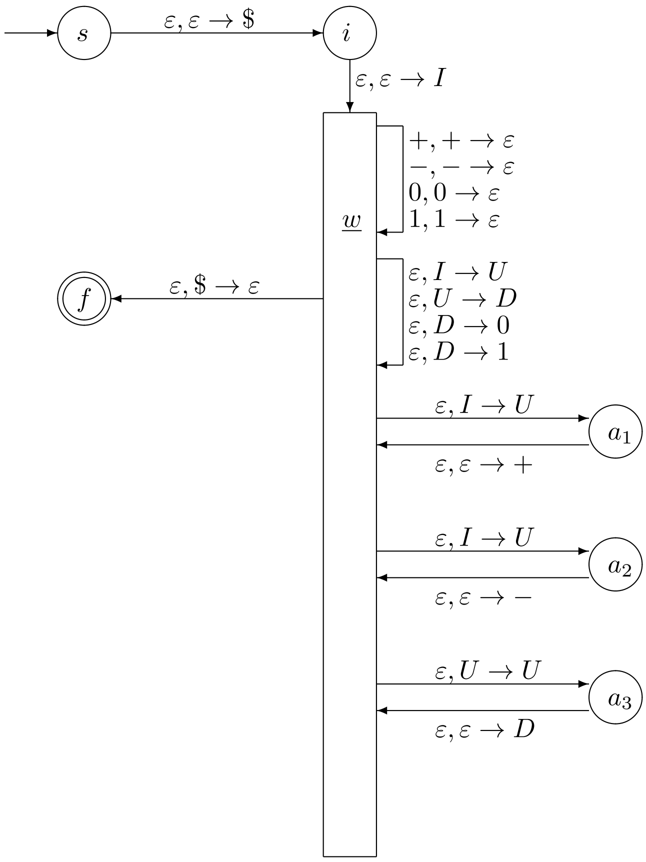
- first, we pop U and push the last symbol of the right-hand side – in this cases, the symbol U into the stack, getting into the auxiliary state a_3;
- finally, we push D into the stack, and go back to the working state w.

Let us illustrate this step by step.

First, we pop U, push U, and go into the state a_3:



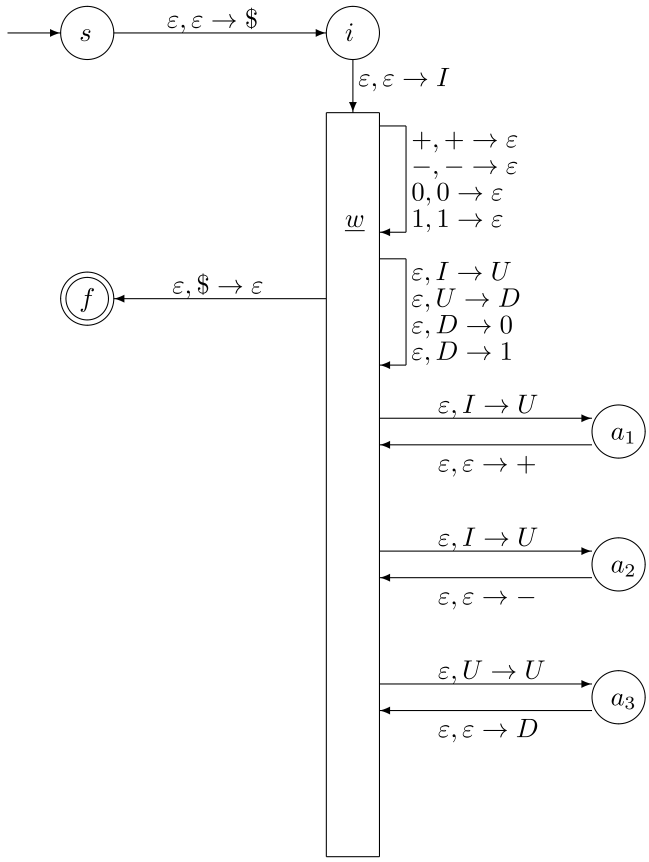
The stack remains unchanged. Next, we push D into the stack and get back to the working state:



The stack will now have D on top:

D
U
\$

Then, we use the rule $D \to 1$, which corresponds to the rule $\varepsilon, D \to 1$ of the pushdown automaton. Namely, we replace D on top of the stack with 1.



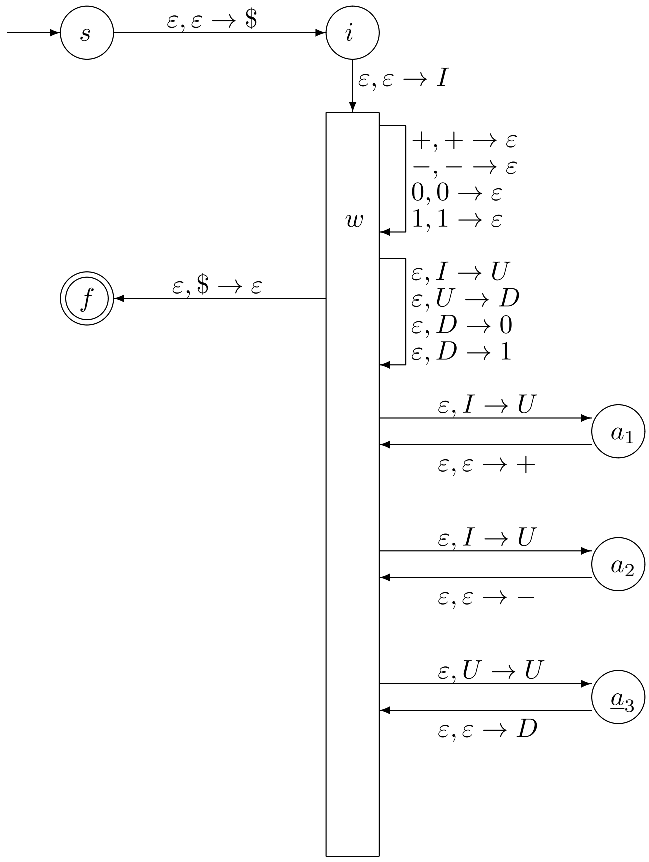
Now, the stack has the form:

1
U
$

On top of the stack is a terminal symbol 1. The only way to delete it from the stack is to use the rule $1, 1 \to \varepsilon$, i.e., to read symbol 1 and pop 1 from the top of the stack. We still remain in the working state, but the stack changes to

U
$

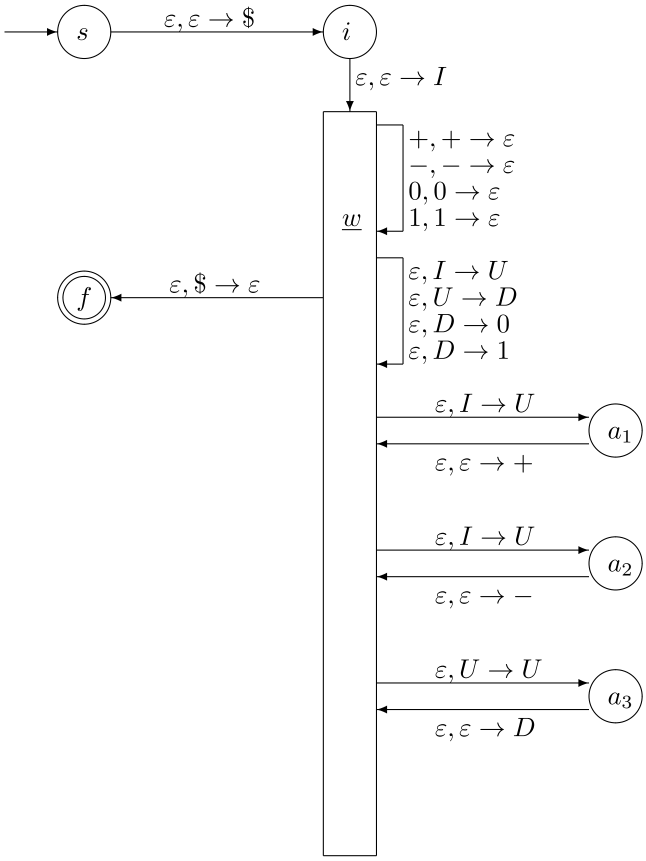
Next, we again use the rule $U \to DU$ of the grammar. So, first, we replace U on top of the stack with U and go to state a_3:



The stack takes the following form:

U
$\$$

Next, we push D into the stack and go back to the working state:



The stack now has the form:

D
U
\$

Next, we use the rule $D \to 1$ from the grammar, which corresponds to the rule $\varepsilon, D \to 1$ of the pushdown automaton. We replace D on top of the stack with 1, and stay in the same state w. The stack now takes the following form:

1
U
\$

On top of the stack is a terminal symbol 1. So eliminate this symbol from the stack, we use the rule $1, 1 \to \varepsilon$, i.e., we read the symbol 1 and delete 1 from the top of the stack. Then, the stack takes the following form:

U
\$

After this, we use the rule $U \to D$ of the grammar, that corresponds to the rule $\varepsilon, U \to D$ of the pushdown automaton. Namely, we replace U on top of the stack with D. Thus, the stack takes the following form:

D
\$

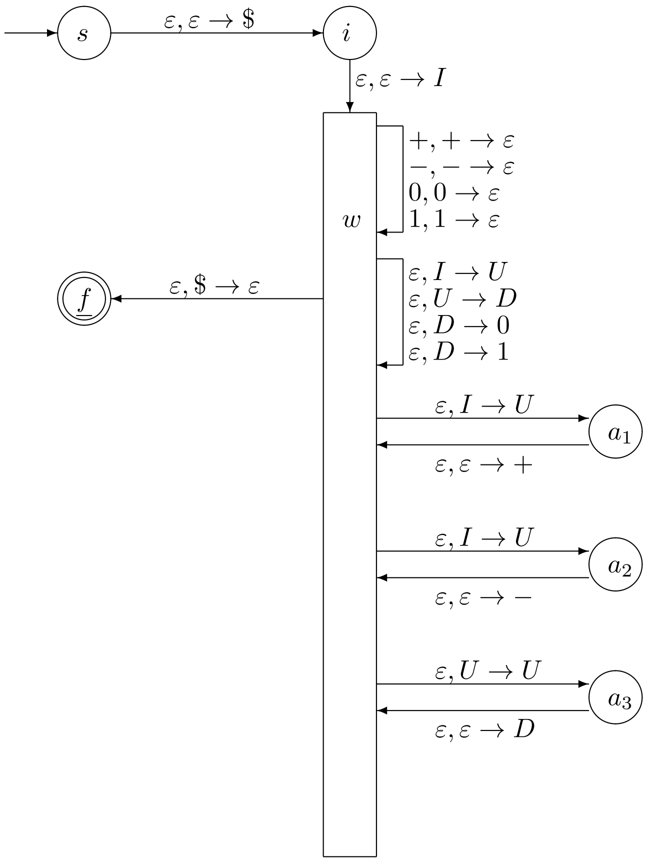
Then, we use the rule $D \to 0$ of the grammar, that corresponds to the rule $\varepsilon, D \to 0$ of the pushdown automaton. Namely, we replace D on top of the stack with 0. Thus, the stack takes the following form:

0
\$

On top of the stack is a terminal symbol 0. So eliminate this symbol from the stack, we use the rule $0, 0 \to \varepsilon$, i.e., we read the last symbol 0 of the word and delete 0 from the top of the stack. Then, the stack takes the following form:

\$

We have read all the symbols of the original word, and we only have the dollar sign remaining in the stack. Thus, we can use the rule $\varepsilon, \$ \to \varepsilon$ to delete the dollar sign from the stack and to move to the final state f:



The stack is now empty. We have read all the symbols of the given word and we end up in the final state with the empty stack. Thus, the word +110 is accepted.

A graphical description of the transitions.

read						+			
state	s	i	w	a_1	w	w	a_3	w	w
stack		\$	I \$	U \$	+ U \$	U \$	U \$	D U \$	1 U \$

read	1				1			0	
state	w	a_3	w	w	w	w	w	w	f
stack	U \$	U \$	D U \$	1 U \$	U \$	D \$	0 \$	\$	