# Solution to Homework Problem 24

**Homework problem 24.** Prove that the cubic root of 18 is not a rational number.

**Solution.** Let us prove it by contradiction. Let us assume that $\sqrt[3]{18}$ is a rational number, i.e., $\sqrt[3]{18} = a/b$ for some integers $a$ and $b$.

If the numbers $a$ and $b$ have a common factor, then we can divide both $a$ and $b$ by this factor and get the same ratio. Thus, we can always find $a$ and $b$ that have no common factors.

Let us now get a contradiction.

- Multiplying both sides of the above equality by $b$, we get $\sqrt[3]{18} \cdot b = a$.
- Cubing both sides, we get $18 \cdot b^3 = a^3$, i.e., $2 \cdot 3^2 \cdot b^3 = a^3$.
- The left-hand side of this equality is divisible by 3, so the right-hand side $a^3 = a \cdot a \cdot a$ must also be divisible by 3.
- Thus, $a$ is divisible by 3, i.e., $a = 3 \cdot p$ for some integer $p$.
- For $a = 3 \cdot p$, we have $a^3 = (3 \cdot p) \cdot (3 \cdot p) \cdot (3 \cdot p) = 3^3 \cdot p^3$.
- Substituting $a^3 = 3^3 \cdot p^3$ into the formula $2 \cdot 3^2 \cdot b^3 = a^3$, we get $2 \cdot 3^2 \cdot b^3 = 3^3 \cdot p^3$.
- Dividing both sides by $3^2$, we get $2 \cdot b^3 = 3 \cdot p^3$.
- The right-hand side of this equality is divisible by 3, so the left-hand side $2 \cdot b^3 = 2 \cdot 2 \cdot 2 \cdot b \cdot b \cdot b$ must also be divisible by 3.
- Thus, $b$ is divisible by 3.
- So, $a$ and $b$ have a common factor 3 – which contradicts to the fact that $a$ and $b$ have no common factors.

This contradiction shows that our original assumption – that $\sqrt[3]{18}$ is a rational number – is wrong. The statement is proven.