Solution to Homework 5

Task: Suppose that someone marks every time the price of a share increases (I) by \$1 and decreases (D) by \$1. As a result, you get a sequence like $IDII$. Based on the sequence, we want to detect whether eventually, the price increased, i.e., whether the sequence contained more increases than decreases. Prove that the language L of all the sequences corresponding to eventual increase is not regular.

Solution. We will prove this result by contradiction. Let us assume that the language L is regular, and let us show that this assumption leads to a contradiction.

Since this language is regular, according to the Pumping Lemma, there exists an integer p such that every word from L whose length $\text{len}(w)$ is at least p can be represented as a concatenation $w = xyz$, where:

- y is non-empty;
- the length $\text{len}(xy)$ does not exceed p, and
- for every natural number i, the word $xy^iz \stackrel{\text{def}}{=} xy\ldots yz$, in which y is repeated i times, also belongs to the language L.

Let us take the word

$$w = D^pI^{p+1} = D\ldots DI\ldots I,$$

in which first the letter D is repeated p times and then the letter I is repeated $p+1$ times. The length of this word is $p+p+1 = 2p+1 > p$. So, by pumping lemma, this word can be represented as $w = xyz$ with $\text{len}(xy) \le p$. The word $w = xyz$ starts with xy, and the length of xy is smaller than or equal to p. Thus, xy is among the first p symbols of the word w – and these symbols are all D's. So, the word y only has D's.

In the original word $w = xyz$, we had D repeated p times and I repeated $p+1$ times. When we go from the word $w = xyz$ to the word $xyyz$, we add D's, and we do not add any I's. Thus, we still have I repeated $p+1$ times, but the number of times D is repeated is now larger than p, i.e., larger than or equal to $p+1$. In any word from the language L, we should have more I's than D's. In the word $xyyz$, this condition is not satisfied. Thus, the word $xyyz$ cannot be in the language L.

On the other hand, by Pumping Lemma, the word $xyyz$ must be in the language L. So, we proved two opposite statements:

- that this word *is not* in L and
- that this word *is* in L.

This is a contradiction.

The only assumption that led to this contradiction is that L is a regular language. Thus, this assumption is false, so the language L is not regular.