Solutions to Test 1

Problem 1. Why do we need to study automata? Provide two main reasons.

Solution to Problem 1.

- To help develop a general understanding of which general problems are solvable and which are not.
- To understand how program are compiled.

Problem 2–4. Let us consider the automaton that has two states: n (the student is in normal mood) and h (the student is happy); n is the starting state, h is the final state. The only two symbols are g (the student got a good grade on this test) and t (time passed).

- From n, g leads to h, and t leads back to n.
- From h, g leads back to h and t leads to n.

Problem 2. Trace, step-by-step, how this finite automaton will check that the word gtg belongs to this language. Use the tracing to find the parts x, y, and z of the word gtg corresponding to the Pumping Lemma. Check that the "pumped" word $xyyz$ will also be accepted by this automaton.

Solution to Problem 2. Let us first trace how the automaton will accept the word gtg:

- we start in the starting state n;
- we read the first symbol g and move to h;
- we read t and move back to n;
- we read g and go back to h.

We have read all the letters of the word, we are in the final state, so the word is accepted.

Let us now trace how the automaton will accept the word gtg:

$$\begin{array}{ccccccc} | & g & | & t & | & g & | \\ \underline{n} & & h & & \underline{n} & & h \end{array}$$

In this derivation, the first pair of repeating states is the pair of the n states: So:

- x is what is before the first repetition, i.e., $x = \Lambda$;
- y is what is in between the repetitions, i.e., $y = gt$; and
- z is what is after the second repetition, i.e., $z = g$.

By repeating the part between the two repetitions we get the derivation of the word $xyyz = gtgtg$:

$$\begin{array}{ccccccccccc} | & g & | & t & | & g & | & t & | & g & | \\ n & & h & & n & & h & & n & & h \end{array}$$

Problem 3. Write down the tuple $\langle Q, \Sigma, \delta, q_0, F\rangle$ corresponding to this automaton:

- Q is the set of all the states,
- Σ is the alphabet, i.e., the set of all the symbols that this automaton can encounter;
- $\delta : Q \times \Sigma \to Q$ is the function that describes, for each state q and for each symbol s, the state $\delta(q, s)$ to which the automaton that was originally in the state q moves when it sees the symbol s (you do not need to describe all possible transitions this way, just describe two of them);
- q_0 is the staring state, and
- F is the set of all final states.

Solution to Problem 3. Here, $Q = \{n, h\}$, $\Sigma = \{g, t\}$, $q_0 = n$, $F = \{h\}$, and the function δ is described by the following table:

	n	h
g	h	h
t	n	n

Problem 4. Use a general algorithm that we had in class to generate a context-free grammar corresponding to this automaton. Show how this grammar will generate the word gtg.

Solution to Problem 4. The corresponding grammar has variables N and H corresponding to the states of the automaton. The variable N corresponding to the starting state n is the starting variable. We have the following rules:

$$N \to gH;$$

$$N \to tN;$$

$$H \to gH;$$

$$H \to tN;$$

$$H \to \varepsilon.$$

The corresponding derivation is:

$$\underline{N} \to g\underline{H} \to gt\underline{N} \to gtg\underline{H} \to gtg.$$

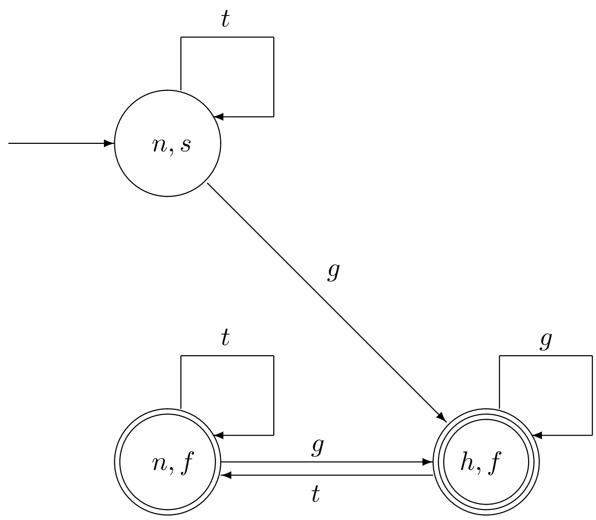
Problem 5. Let A_1 be the automaton described in Problem 2. Let A_2 be an automaton that accepts all the strings that contain at least one symbol g – indicating good news. This automaton has two states: the starting state s, and the final state f. The transitions are as follows:

- from the start state, t lead back to the start state, while g leads to the final state f;
- from the final state f, any symbol leads back to this state.

Use the algorithm that we had in class to describe the following two new automata:

- the automaton that recognizes the union $A_1 \cup A_2$ of the two corresponding languages, and
- the automaton that recognizes the intersection of the languages A_1 and A_2.

Solution to Problem 5.

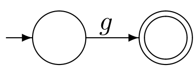


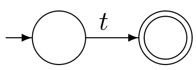
Problem 6. Use the general algorithm that we learned in class to design a non-deterministic finite automaton that recognizes the language $(g \cup t)^* g$:

- first, describe the automata for recognizing g and t;
- then, combine them into the automata for recognizing the union $g \cup t$, and the Kleene star $(g \cup t)^*$;
- finally, combine the automata for $(g \cup t)^*$ and g into an automaton for recognizing the desired composition of the two languages.

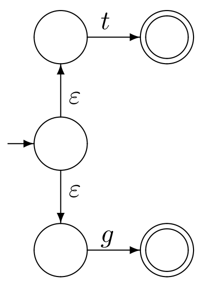
Solution to Problem 6. We start with the standard non-deterministic automata for recognizing:

- the language g – that consists of a single word g, and
- the language t – that consists of a single word t:

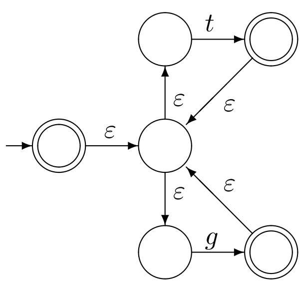




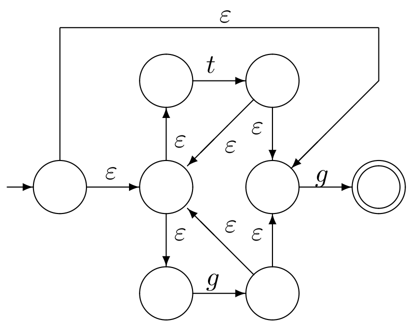
Then, we use the general algorithm for the union to design a non-deterministic automaton for recognizing the language $g \cup t$:



Now, we apply a standard algorithm for the Kleene star, and we get the following non-deterministic automaton for $(g \cup t)^*$:

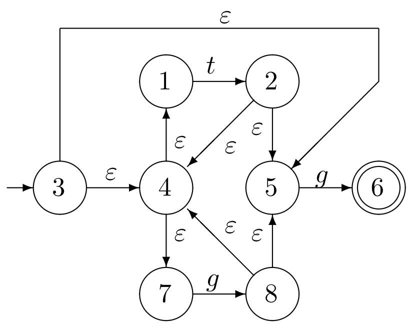


Now, we use the algorithm for concatenation for combine them: final states of the automaton for $(g \cup t)^*$ are no longer final, and from each of them, we add a jump to the starting state of the automaton for g:



Problem 7. Use the general algorithm to transform the resulting non-deterministic finite automaton into a deterministic one.

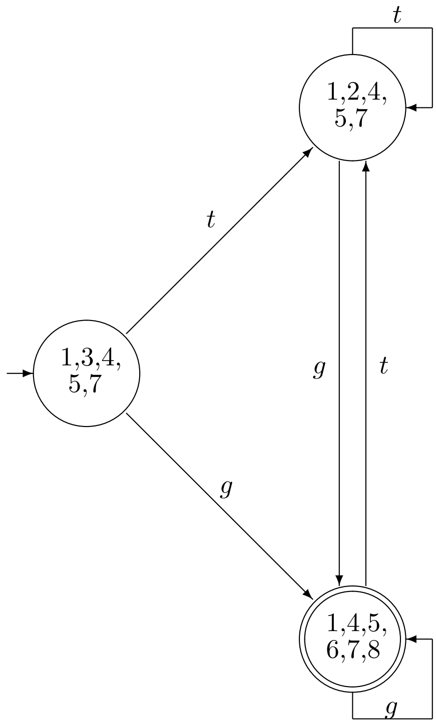
Solution to Problem 7. Let us first enumerate the states of the resulting non-deterministic automaton.



In the beginning, before we see any symbol, we are in state 3, and we can jump to 1, 4, 5, and 7. So, the resulting state is $\{1,3,4,5,7\}$.

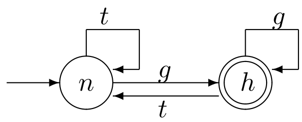
- If in the state $\{1,3,4,5,7\}$, we see letter t, we can go to 2 and from there, jump to 1, 4, 5, and 7. Thus, the resulting state is $\{1,2,4,5,7\}$.
- If in the state $\{1,3,4,5,7\}$, we see letter g, we can go to 8 and from there, jump to 1, 4, 5, and 7. We can also go to 6. Thus, the resulting state is $\{1,4,5,6,7,8\}$. This state contains the final state 6 and is, thus, final.
- If in the state $\{1,2,4,5,7\}$, we see letter t, we can go to 2 and from there, jump to 1, 4, 5, and 7. Thus, the resulting state is the same state $\{1,2,4,5,7\}$.
- If in the state $\{1,2,4,5,7\}$, we see letter g, we can go to 8 and from there, jump to 1, 4, 5, and 7. We can also go to 6. Thus, the resulting state is $\{1,4,5,6,7,8\}$.
- If in the state $\{1,4,5,6,7,8\}$, we see letter t, we can go to 2 and from there, jump to 1, 4, 5, and 7. Thus, the resulting state is $\{1,2,4,5,7\}$.
- If in the state $\{1,4,5,6,7,8\}$, we see letter g, we can go to 8 and from there, jump to 1, 4, 5, and 7. We can also go to 6. Thus, the resulting state is the same state $\{1,4,5,6,7,8\}$.

Thus, we arrive at the following deterministic automaton.


t
1,2,4,
5,7
t
1,3,4,
5,7
g
t
g
1,4,5,
6,7,8
g

Problem 8–9. Use a general algorithm to transform the finite automaton from Problem 2 into the corresponding regular expression. Start with eliminating the state n.

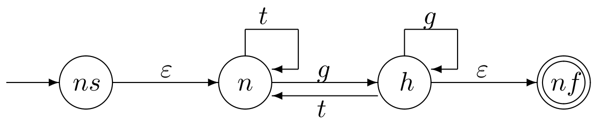
Solution to Problem 8–9. We start with the described automaton:



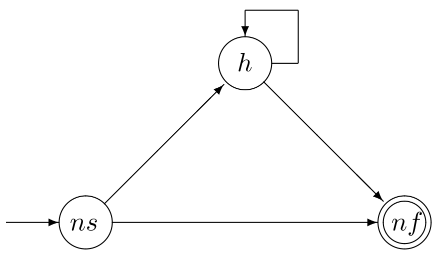
According to the general algorithm, first we add a new start state ns and a new final state f, and we add jumps:

- from the new start state ns to the old start state, and
- from each old final state to the new final state nf.

As a result, we get the following automaton.



Then, we need to eliminate the two intermediate states n and g one by one. We first eliminate the state n. First, we draw all possible arrows:



Now, to find expressions to place at all these arrows, we will use the general formula

$$R'_{i,j} = R_{i,j} \cup (R_{i,k} R^*_{k,k} R_{k,j}),$$

where k is the state that we are eliminating, i.e., in this case, the state $k = n$.

By applying this formula, and by using simplification formulas described in the lecture, we get the following results:

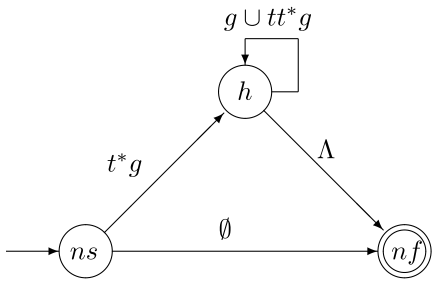
$$R'_{ns,h} = R_{ns,h} \cup (R_{ns,n} R^*_{n,n} R_{n,h}) = \emptyset \cup (\Lambda t^* g) = \emptyset \cup t^* g = t^* g;$$

$$R'_{ns,nf} = R_{ns,nf} \cup (R_{ns,n} R^*_{n,n} R_{n,nf}) = \emptyset \cup (\Lambda t^* \emptyset) = \emptyset \cup \emptyset = \emptyset;$$

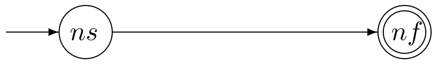
$$R'_{h,h} = R_{h,h} \cup (R_{h,n} R^*_{n,n} R_{n,h}) = g \cup (tt^* g);$$

$$R'_{n,nf} = R_{h,nf} \cup (R_{h,n} R^*_{n,n} R_{n,nf}) = \Lambda \cup (tt^* \emptyset) = \Lambda \cup \emptyset = \Lambda.$$

Thus, the 3-state a-automaton takes the following form:



Now, all that remains to do is to go from here to the 2-state a-automaton by eliminating the remaining state h:



The final expression is the corresponding expression for $R'_{ns,nf}$:

$$R'_{ns,nf} = R_{ns,nf} \cup (R_{ns,h} R^*_{h,h} R_{n,nf}) =$$

$$\emptyset \cup (t^* g (g \cup tt^* g)^* \Lambda) = \emptyset \cup (t^* g (g \cup tt^* g)^*) =$$

$$t^* g (g \cup tt^* g)^*.$$

The formula on the previous line is a regular expression corresponding to the original automaton.

Problem 10. Prove that the language L of all the words that have more g's than t's is not regular.

Solution to Problem 10. We will prove it by contradiction. Let us assume that the language L is regular, and let us show that this assumption leads to a contradiction.

Since this language is regular, according to the Pumping Lemma, there exists an integer p such that every word from L whose length $\text{len}(w)$ is at least p can be represented as a concatenation $w = xyz$, where:

- y is non-empty;
- the length $\text{len}(xy)$ does not exceed p, and
- for every natural number i, the word $xy^iz \stackrel{\text{def}}{=} xy\ldots yz$, in which y is repeated i times, also belongs to the language L.

Let us take the word

$$w = t^p g^{p+1} = t\ldots tg\ldots g,$$

in which first t is repeated p times, then g is repeated $p+1$ times. The length of this word is $p + p + 1 = 2p + 1 > p$. So, by pumping lemma, this word can be represented as $w = xyz$ with $\text{len}(xy) \le p$. This word starts with xy, and the length of xy is smaller than or equal to p. Thus, xy is among the first p symbols of the word w – and these symbols are all t's. So, the word y only has t's.

Thus, when we go from the word $w = xyz$ to the word $xyyz$, we add at least one t, and we do not add any g's. So, in the word $xyyz$, there are now at least $p+1$ letters t. Since there are at $p+1$ letters g, this means that the number of t's is no longer larger than the number of g's. Thus, the word $xyyz$ cannot be in the language L, since by definition L only contains words which have more g's than t's.

On the other hand, by Pumping Lemma, the word $xyyz$ must be in the language L. So, we proved two opposite statements:

- that this word *is not* in L and
- that this word *is* in L.

This is a contradiction.

The only assumption that led to this contradiction is that L is a regular language. Thus, this assumption is false, so L is not regular.