

Solution to Problem 13

Task. Three chess players a , b , and c of almost equal strength play in a tournament. After each game, we record who was the winner. The number of games won by a is exactly 1 larger than the number of games one by b , and the number of games won by c is 1 smaller than the number of games won by b . An example of such sequence is $abcaba$: a won 3 games, b won $3 - 1 = 2$ games, and c won $2 - 1 = 1$ game. Prove that the language of all such sequences is not context-free.

Solution. Let us prove that this language is not context-free.

The proof will be by contradiction. Let us assume that this language is context-free. Then, by pumping lemma, there exists an integer p such that every word w from the language L whose length is at least p can be represented as $w = uvxyz$, where:

- $\text{len}(vxy) \leq p$;
- $\text{len}(vy) > 0$, and
- for all natural numbers i , the word $uv^i xy^i z$ also belongs to the language L .

Let us take the word

$$w = a^{p+2}b^{p+1}c^p = a \dots ab \dots bc \dots c,$$

where a is repeated $p + 2$ times, b is repeated $p + 1$ times, and c is repeated p times. The length of this word – i.e., the number of symbols in this word – is equal to $p + 2 + p + 1 + p = 3p + 3$. Clearly, $3p + 3 \geq p$, so, according to the Pumping Lemma, this word can be described as $uvxyz$ with the above properties.

Where can the central part vxy of this word be? We know that the length $\text{len}(vxy)$ of this part cannot exceed p . Thus, it cannot contain three different types of symbols: a 's, b 's, and c 's – since then it would have to include all $p + 1$ symbols b plus additional a and c symbols, so its length would have been larger than p . So, there are only 5 cases remaining for the location of the part vxy :

1. it can be in the a 's;
2. it can be in a 's and b 's;
3. it can be in b 's;

4. it can be in b 's and c 's;
5. it can be in c 's.

Let us consider these cases one by one.

Case 1. If vxy is in the a 's, this means that the parts v and y contain only a 's. Thus, when we pump, i.e., when we go from the original word $uvxyz$ to the word $uv^2xy^2z = uvvxyyz$, we add a 's – but we do not add any b 's or c 's. In the original word $w = a^{p+2}b^{p+1}c^p$, there was a balance between letters of the three types. When we add more a 's, the balance is disrupted: the difference between number of a 's and number of b 's is now larger than 1. Since the language L only contains the words for which the above proportions have to be satisfied, the word $uvvxyyz$ cannot belong to the language L .

Case 2. If vxy is in a 's and in b 's, this means that the parts v and y contain only a 's and b 's. Thus, when we pump, i.e., when we go from the original word $uvxyz$ to the word $uv^2xy^2z = uvvxyyz$, we add a 's and b 's – but we do not add any c 's. In the original word $w = a^{p+2}b^{p+1}c^p$, there was the desired balance between numbers of letters of all three types. When we add more a 's and/or b 's, the balance is disrupted, so the word $uvvxyyz$ cannot belong to the language L .

Similarly, we can see that in the other 3 cases, we also get a contradiction. This means that the original assumption – that the language L is context-free – is wrong. Thus, the language L is not context-free.