Solution to Homework Problem 20

**Homework Problem 20.** As we discuss in class, a Turing machine can be described as a finite automata with two stacks:

- the right stack contains, on top, the symbol to which the head points; below is the next symbol to the right, then the next to next symbol to the right, etc.;
- the left stack contains, on top, the symbol directly to the left of the head (if there is a one), under it is the next symbol to the left, etc.

On the example a Turing machine that computes $n - 1$ for a binary number $n = 2$, show, step-by-step:

- how each state of the corresponding Turing machine can be represented in terms of two stacks, and
- how each transition from one state to another can be implemented by push and pop operations.

**Solution.** The rules of this Turing machine are as follows:

start, $-$ $\rightarrow$ moving, R
moving, 0 $\rightarrow$ 1, R
moving, 1 $\rightarrow$ 0, L, back
back, 0 $\rightarrow$ L
back, 1 $\rightarrow$ L
back, $-$ $\rightarrow$ halt

1. At first, we have the following configuration:

```
|   | 0 | 1 |   |   |   | ... |
```
start

Here, the left stack is empty, and the right stack has the form

```
|   | 0 | 1 |
```

2. Then, the configuration changes to:

```
|   | 0 | 1 |   |   |   | ... |
```
moving

Here, the two stacks have the following form:
To get to this configuration, we pop the symbol \( \text{–} \) (meaning black space) from the right stack and push it into the left stack.

3. Then, the configuration changes to:

\[
-1 \ 1 \ \text{–} \ \text{–} \ \text{–} \ \text{–} \ \text{–} \ \cdots \ \text{moving}
\]

Here, the left stack has the following form:

\[
\begin{array}{c}
1 \\
\end{array}
\]

and the right stack has the following form:

\[
\begin{array}{c}
1 \\
\end{array}
\]

To get to this configuration, we pop 1 from the right stack, change it to 0, and push it into the left stack.

4. Then, the configuration changes to:

\[
-1 \ 0 \ \text{–} \ \text{–} \ \text{–} \ \text{–} \ \text{–} \ \cdots \ \text{back}
\]

Here, the left stack has the following form:

\[
\begin{array}{c}
\text{–} \\
\end{array}
\]

and the right stack has the following form:

\[
\begin{array}{c}
1 \\
0 \\
\end{array}
\]

To get to this configuration, we replace 1 with 0, pop 0 from the right stack and push it into the left stack.

5. Then, the configuration changes to:

\[
-1 \ 0 \ \text{–} \ \text{–} \ \text{–} \ \text{–} \ \cdots \ \text{back}
\]

Here, the left stack is empty, and the right stack has the following form:

\[
\begin{array}{c}
\text{–} \\
1 \\
0 \\
\end{array}
\]

To get to this configuration, we pop blank from the left stack and push it into the right stack.

6. Then, the configuration changes to:

\[
-1 \ 0 \ \text{–} \ \text{–} \ \text{–} \ \text{–} \ \cdots \ \text{halt}
\]

Here, we did not change anything on the tape, and we did not change the location of the head, so the stacks remain the same.