Solution to Problem 13

Task. To properly set up a table for a celebratory lunch, for each dinner plate (d), a workstudy needs to have exactly two folks (f) and exactly two spoons (s). For example, we can have a sequence $f\,d\,f\,f\,s\,s\,s\,d\,f\,s$, while a sequence $d\,f\,f\,f\,s\,s$ is not good since it contains an extra fork. Prove that the language of all such sequences is not context-free.

Solution. Let us prove that this language is not context-free.

The proof will be by contradiction. Let us assume that this language is context-free. Then, by pumping lemma, there exists an integer p such that every word w from the language L whose length is at least p can be represented as $w = uvxyz$, where:

- $\text{len}(vxy) \le p$;
- $\text{len}(vy) > 0$, and
- for all natural numbers i, the word uv^ixy^iz also belongs to the language L.

Let us take the word

$$w = d^p f^{2p} s^{2p} = d \ldots d\, f \ldots f s \ldots s,$$

where d is repeated p times, f is repeated $2p$ times, and s is repeated $2p$ times. The length of this word – i.e., the number of symbols in this word – is equal to $p + 2p + 2p = 5p$. Clearly, $5p \ge p$, so, according to the Pumping Lemma, this word can be described as $uvxyz$ with the above properties.

Where can the central part vxy of this word be? We know that the length $\text{len}(vxy)$ of this part cannot exceed p. Thus, it cannot contain three different types of symbols: d's, f's, and s's – since then it would have to include all $2p$ symbols f plus additional d and s symbols, so its length would have been larger than p. So, there are only 5 cases remaining for the location of the part vxy:

1. it can be in the d's;
2. it can be in d's and f's;
3. it can be in f's;
4. it can be in f's and s's;
5. it can be in s's.

Let us consider these cases one by one.

Case 1. If vxy is in the d's, this means that the parts v and y contain only d's. Thus, when we pump, i.e., when we go from the original word $uvxyz$ to the word $uv^2xy^2z = uvvxyyz$, we add d's – but we do not add any f's or s's. In the original word $w = d^p f^{2p} s^{2p}$, there was a balance between letters of the three types. When we add more d's, the balance is disrupted: the ratio between number of f's and number of d's is now smaller than 2. Since the language L only contains the words for which the above proportions have to be satisfied, the word $uvvxyyz$ cannot belong to the language L.

Case 2. If vxy is in d's and in f's, this means that the parts v and y contain only d's and f's. Thus, when we pump, i.e., when we go from the original word $uvxyz$ to the word $uv^2xy^2z = uvvxyyz$, we add d's and f's – but we do not add any s's. In the original word $w = d^p f^{2p} s^{2p}$, there was the desired balance between numbers of letters of all three types. When we add more d's and/or f's, the balance is disrupted, so the word $uvvxyyz$ cannot belong to the language L.

Similarly, we can see that in the other 3 cases, we also get a contradiction. This means that the original assumption – that the language L is context-free – is wrong. Thus, the language L is not context-free.