Solution to Homework 5

Task: Several students forgot their laptops at home, they asked CS IT folks. CS IT has several laptops (L) and several chargers (C). It would be nice to have a charger for each laptop. Prove that the following language is not regular: the set S of all sequences of Ls and Cs in which there are at least as many C's as L's. For example, the word LCCLC is in the language S, while the word CLL is not in S.

Solution. We will prove this result by contradiction. Let us assume that the language S is regular, and let us show that this assumption leads to a contradiction.

Since this language is regular, according to the Pumping Lemma, there exists an integer p such that every word from S whose length $\text{len}(w)$ is at least p can be represented as a concatenation $w = xyz$, where:

- y is non-empty;
- the length $\text{len}(xy)$ does not exceed p, and
- for every natural number i, the word $xy^iz \stackrel{\text{def}}{=} xy\ldots yz$, in which y is repeated i times, also belongs to the language S.

Let us take the word

$$w = \text{L}^p\text{C}^p = \text{L}\ldots\text{LC}\ldots\text{C},$$

in which first the symbol L is repeated p times and then the symbol C is repeated p times. The length of this word is $p + p = 2p > p$. So, by pumping lemma, this word can be represented as $w = xyz$ with $\text{len}(xy) \le p$. The word $w = xyz$ starts with xy, and the length of xy is smaller than or equal to p. Thus, xy is among the first p symbols of the word w – and these symbols are all Ls. So, the word y only has Ls.

In the original word $w = xyz$, we had L repeated p times and C repeated p times. When we go from the word $w = xyz$ to the word $xyyz$, we add Ls, and we do not add any Cs. Thus, we still have C repeated p times, but the number of times L is repeated is now larger than p. In any word from the language S, we should have at least as many Cs as Ls. In the word $xyyz$, this condition is not satisfied. Thus, the word $xyyz$ cannot be in the language S.

On the other hand, by Pumping Lemma, the word $xyyz$ must be in the language S. So, we proved two opposite statements:

- that this word *is not* in S and
- that this word *is* in S.

This is a contradiction.

The only assumption that led to this contradiction is that S is a regular language. Thus, this assumption is false, so the language S is not regular.