Solution to Homework Problem 24

Homework problem 24. Prove that the cubic root of 54 is not a rational number.

Solution. Let us prove it by contradiction. Let us assume that $\sqrt[3]{54}$ is a rational number, i.e., $\sqrt[3]{54} = a/b$ for some integers a and b.

If the numbers a and b have a common factor, then we can divide both a and b by this factor and get the same ratio. Thus, we can always find a and b that have no common factors.

Let us now get a contradiction.

- Multiplying both sides of the above equality by b, we get $\sqrt[3]{54} \cdot b = a$.
- Cubing both sides, we get $54 \cdot b^3 = a^3$, i.e., $2 \cdot 3^3 \cdot b^3 = a^3$.
- The left-hand side of this equality is divisible by 2, so the right-hand side $a^3 = a \cdot a \cdot a$ must also be divisible by 2.
- Since 2 is a prime number, this means that one of the factors in the product $a \cdot a \cdot a$ – i.e., number a – is divisible by 2, i.e., $a = 2 \cdot p$ for some integer p.
- For $a = 2 \cdot p$, we have $a^3 = (2 \cdot p) \cdot (2 \cdot p) \cdot (2 \cdot p) = 2^3 \cdot p^3$.
- Substituting $a^3 = 2^3 \cdot p^3$ into the formula $2 \cdot 3^3 \cdot b^3 = a^3$, we get $2 \cdot 3^3 \cdot b^3 = 2^3 \cdot p^3$.
- Dividing both sides by 2, we get $3^3 \cdot b^3 = 2^2 \cdot p^3$.
- The right-hand side of this equality is divisible by 2, so the left-hand side $3^3 \cdot b^3 = 3 \cdot 3 \cdot 3 \cdot b \cdot b \cdot b$ must also be divisible by 2.
- Thus, one of the factors in the right-hand side is divisible by 2.
- Since 3 is not divisible by 2, thus b is divisible by 2.
- So, a and b have a common factor 2 – which contradicts to the fact that a and b have no common factors.

This contradiction shows that our original assumption – that $\sqrt[3]{54}$ is a rational number – is wrong. The statement is proven.