Solution to Homework 18

Question. Describe how to use linearization to estimate the uncertainty of the value $y = f(x_1, x_2) = x_1^2 + x_2^2$ when the measurement results are $x_1 = 1$ and $x_2 = 2$ and the upper bounds on the uncertainties are $\Delta_1 = 0.1$ and $\Delta_2 = 0.2$. Use both actual values of the partial derivatives and the values obtained by numerical differentiation.

Answer. The general formula for the resulting upper bound on the absolute value of the estimation error is

$$\Delta = \sum_{i=1}^{n} |c_i| \cdot \Delta_i,$$

where c_i is an estimation of the i-th partial derivative. In our case,

$$\Delta = |c_1| \cdot \Delta_1 + |c_2| \cdot \Delta_2.$$

If we actually compute the corresponding partial derivatives, we get the following values:

$$c_1 = \frac{\partial f}{\partial x_1} = 2x_1 \text{ and } c_2 = \frac{\partial f}{\partial x_2} = 2x_2.$$

In our case, $x_1 = 1$ and $x_2 = 2$, so $c_1 = 2 \cdot 1 = 2$ and $c_2 = 2 \cdot 2 = 4$. For these values c_i, we have:

$$\Delta = |2| \cdot 0.1 + |4| \cdot 0.2 = 0.2 + 0.8 = 1.0.$$

Numerical differentiation means estimating each derivative as

$$c_i \approx \frac{f(x_1, \ldots, x_{i-1}, x_i + \Delta_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta_i}.$$

In our case:

$$f(x_1, x_2) = x_1^2 + x_2^2 = 1^2 + 2^2 = 1 + 4 = 5$$

and

$$f(x_1 + \Delta_1, x_2) = (x_1 + \Delta_1)^2 + x_2^2 = (1 + 0.1)^2 + 2^2 = 1.1^2 + 2^2 =$$

$$1.21 + 4 = 5.21.$$

Thus,

$$c_1 = \frac{f(x_1 + \Delta_1, x_2) - f(x_1, x_2)}{\Delta_1} = \frac{5.21 - 5}{0.1} = \frac{0.21}{0.1} = 2.1.$$

Similarly:

$$f(x_1, x_2 + \Delta_2) = x_1^2 + (x_2 + \Delta_2)^2 = 1^2 + (2 + 0.2)^2 = 1^2 + 2.2^2 =$$

$$1 + 4.84 = 5.84.$$

Thus,

$$c_2 = \frac{f(x_1, x_2 + \Delta_2) - f(x_1, x_2)}{\Delta_2} = \frac{5.84 - 5}{0.2} = \frac{0.84}{0.2} = 4.2.$$

For these approximate values of the partial derivatives, we get:

$$\Delta = |c_1| \cdot \Delta_1 + |c_2| \cdot \Delta_2 = |2.1| \cdot 0.1 + |4.2| \cdot 0.2 = 0.21 + 0.84 = 1.05.$$