What is subjective probability?

Question. How can we describe a person's degree of confidence in numerical terms?

Analysis of the problem. To answer this question, we need to take into an account that the ultimate goal is always to make decisions. So, let us recall how can we make decisions in situations when we know the actual – objective – probability p of some event E – e.g., the probability that there will be rain tomorrow. From the commonsense viewpoint, the probability is, in effect, a frequency: if something happened 300 times out of 1000, we can that the probability is $300/1000 \approx 0.3$. Of course, it is not necessaruly equal: e.g., the probability of a fair coin falling head is 0.5, but out of 11 tries, we cannot have frequency ot be exactly 0.5 – this w=ould be mean the coin fell head 5.5 times, which makes no sense.

In general, the probability p means that for large N, out of N similar situations, the number of time M that this event happens will satisfy the condition $M/N \approx p$. This means that in N situations, this event will occur is approximately $M \approx p \cdot N$ times.

If someone asks you to buy an option in which you get \$1 if the event E happens and 0 if this event does happenm, what is the amount a that you are willing to pay for this option? If you encounter N such situations and pay a each time, then overall you paid the amount $a \cdot N$. Out of these N situations, the event happens approximately $p \cdot N$ times. So, the fair price is when what you pay is equal to the aount that you gain, i.e., if $a \cdot N = p \cdot N$.

Indeed, if you gain less that what you paid, i.e., if $a \cdot N$ is smaller than $p \cdot N$, then you lose money, so you do not want to participate in such option. On the other hand, if you gain more than what you paid, this means that the proposer of this option will lose money – and why would he or she do that? So, the only fair price is when these two amounts are equal: $a \cdot N = p \cdot N$. Ifw e divide both sides by N, we conclude that $a = p$.

So, we conclude that the amount a that you are willing to pay for the option "\$1 is E else 0" is exactly equal to the objective probability p of this event. This is what probability means in terms of decision making.

Natural idea of subjective probability. It is therefore reasonable to describe subjective probability of an event E as the probability p for which the following two options are equivalent to each other:

- in the first option – we will denote it by $L(E)$ – you get \$1 is E happens;

- in the second option – we will denote it by $L(p)$ – you get \$1 with probability p.

How can we determine this probability. A naive idea is to try all possible values of the probability – but there are infinitely many such values on the interval $[0, 1]$, so this will take forever. A better way is to use bisection. The idea is that:

- if $L(E)$ is worse than $L(p)$, this means that the subjective probability of E is smaller than p;
- on the other hand, if $L(E)$ is better than $L(p)$, this means that the subjective probability of E is larger than p.

In the beginning, since the event happens sometimes – but not always – we know that $L(E)$ is better than the option $L(0)$ – in which we get nothing, and worse than the option $L(1)$ – in which we get \$1 no matter what. Thus, originally, all we know is that the desired subjective probability is somewhere between 0 and 1.

In the bisection algorithms, at each stage of bisection, we have an interval $[\underline{p}, \overline{p}]$ that contains the desired probability p. Initially, $\underline{p} = 0$ and $\overline{p} = 1$. Then, we select a midpoint of this interval

$$m = \frac{\underline{p} + \overline{p}}{2}$$

and compare the option $L(E)$ with the option $L(m)$ in which we get \$1 with probability m.

- if $L(E)$ is worse than $L(m)$, this means the subjective probability p is smaller than m, so we conclude that $p \in [\underline{p}, m]$;
- on the other hand, if $L(E)$ is larger than $L(m)$, this means the subjective probability p is larger than m, so we conclude that $p \in [m, \overline{p}]$.

In both cases, we get a new interval containing p whose width is twice smaller. If we repeat this procedure k times, we get the interval containing p whose width is 2^{-k}. For example, for $k = 7$, we can thus get the subjective probability p with accuracy 1%. For rare events, for which the probability is small, we need more iterations – but it is still doable.