Homework 3

CS 5315 (Theory of Computation)

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Prove the following theorem:

THERE EXISTS A COMPUTABLE FUNCTION THAT IS NOT PRIMITIVE RECURSIVE (PR).

It is necessary to find a computable function and prove that it is not PR.

Part 1 Describe a "code" of a PR function f as the unique integer that identifies that PR function f. This code can be obtained by the following steps:

- **a.** Translate the PR function into an expression in terms of 0, π_i^k, σ, $\circ$ and PR,
- **b.** Write this expression in some language that allows the π_i^k, σ, $\circ$ symbols. For example, LaTeX. This will transform the function expression into its ASCII symbols representation,
- **c.** Transform the ASCII symbols into their binary representation,
- **d.** Interpret the sequence of 0's and 1's as an integer c in the binary form, this integer is the code of the PR function.

Part 2 For every integer c that is a code of a PR function, denote the corresponding PR function by $f_c(n)$. This function f_c takes an integer n as input and returns $f_c(n)$. Given c and n, we can compute $f_c(n)$ as follows:

- **a.** The integer c is a binary sequence (of 0's and 1's),
- **b.** Translate this binary sequence into the ASCII symbols representation,
- **c.** Transform ASCII symbols using LaTeX to a series of correct symbols that can be given as input to a PR compiler,
- **d.** Parse tree of the compiler reconstruct the program of f_c which can be run on a computer.

PROOF

Part 1 Define function f formally.

$$f(n) = \begin{cases} f_n(n) + 1 & \text{if a PR compiler verifies that } n \\ & \text{is a valid PR function code} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since f_n is computable, f is also clearly computable.

Part 2 Prove that function f is not PR.

Assume that function f is PR. Therefore, there exists a code c such that for all n, $f(n)$ coincides with $f_c(n)$, or $\exists c \forall n[f(n) = f_c(n)]$. Then, $f(c) = f_c(c)$, when $n = c$.

By definition (1) of function f, $f(c) = f_c(c) + 1$.

Since $f_c(c) = f(c) = f_c(c) + 1$ and $f_c(c) \neq f_c(c) + 1$, we reached a contradiction! This means that $f(n)$ can not be PR.

Therefore, there exists a function f that is not PR (primitive recursive). $\square$