Solution to Homework 30

Homework 30. Similar to what we did in the class, illustrate the general algorithm of reducing NP problems to satisfiability on the example of the following problem:

- given a bit x,
- find a bit y for which the following formula is true: $x \vee \neg y$.

Solution.

Computational device for checking the desired property. In accordance with the above proof, we need to start with a computational device that, given x and y, checks whether $x \vee \neg y$ is true. In the beginning, we have two cells: an x-cell that contains the input bit x and a y-cell which contains the bit y.

We also need a wire to transmit the information. We will thus send the content of the y-cell to the x-cell, and then use the x-cell to compare its original content with what is send by wire. Once the y-signal is sent, we no longer need it, so we can simply erase it (i.e., replace it with 0).

The whole computation process takes 3 moments of time:

- at moment $t = 1$, the x-cell contains x, the y-cell contains y, and the wire is inactive;
- at moment $t = 2$, the x-cell still contains x, the y-cell now contains 0, and the wire transmits the y signal;
- at moment $t = 3$, the x-cell contains 1 if $x \vee \neg y$ and 0 otherwise, the y-cell contains 0, and the wire is again inactive.

Similarly to the example from the handout, in this computations process, we have 3 cells: the x-cell, the y-cell, and the wire. The x-cell has 2 possible states: 0 and 1, so one bit is sufficient to describe its state. According to the general notation, we will denote the state of this bit at moment t by $s_{1,1,t}$. Similarly, to describe the state of the y-cell, we need one bit $s_{2,1,t}$.

The wire can be in 3 possible states: inactive, sending 0, and sending 1. Thus, to describe the state of the wire, we will need 2 bits. Let the first bit describe whether the wire is active or not, and the second bit describe the signal sent via an active wire. So, the state S_3 of the wire is either 00 (inactive), or 10 (sending 0), or 11 (sending 1).

In this case, $S = 3$, and the number of bits B needed to describe the state of each of the cells is $B = 2$.

Corresponding dynamics of states. Let us describe the above computations in terms of changing states.

At the first moment of time, the wire is inactive: $s_{3,1,1} = s_{3,2,1} = 0$.

At the second moment of time, the first cell retains its state, i.e., $s_{1,1,2} = s_{1,1,1}$. The second cell becomes 0: $s_{2,1,2} = 0$. The wire becomes active: $s_{3,1,2} = 1$, and the signal it transmits is exactly the bit originally stored in the y-cell: $s_{3,2,2} = s_{2,1,1}$.

At the third moment of time, the x-cell gets the value 1 if the property $x \vee \neg y$ is true, where:

- x is the same initial x-state (since we did not change it), i.e., $x = s_{1,1,2}$, and
- y is the state passed through the wire, i.e., $y = s_{3,2,2}$.

Thus, $s_{1,1,3} = 1 \Leftrightarrow (s_{1,1,2} \vee \neg s_{3,2,2})$. The y-cell still contains 0: $s_{2,1,3} = 0$, and the wire is again inactive: $s_{3,1,3} = s_{3,2,3} = 0$.

Describing the dynamics in CNF terms. The above formulas have the form $a = 0$, etc., for some variables a. So, to describe the above formulas in the CNF terms, we need to translate the following general formulas into CNF: $a = 0$, $a = 1$, $a = b$, and $a = 1 \Leftrightarrow (b \vee \neg c)$. Once we do that, we will be able to translate specific formulas by plugging the specific name of the variable a into the corresponding CNF formula.

We already know, from the example presented in the handout, that:

- the CNF form of the formula $a = 0$ is $\neg a$;
- the CNF form of the formula $a = 1$ is a; and
- the CNF form of the formula $a = b$ is $(a \vee \neg b) \,\&\, (\neg a \vee b)$.

Let us use the general algorithm to translate the remaining formula $a = 1 \Leftrightarrow (b \vee \neg c)$ into CNF.

Translating $a = 1 \Leftrightarrow (b \vee \neg c)$ into CNF. For the formula $a = 1 \Leftrightarrow (b \vee \neg c)$, the truth tables for formula F itself and for its negation $\neg F$ take the form

a	b	c	F	$\neg F$
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	1	0
1	1	1	1	0

The corresponding DNF form for $\neg F$ is

$$(\neg a \,\&\, \neg b \,\&\, \neg c) \vee (\neg a \,\&\, b \,\&\, \neg c) \vee (\neg a \,\&\, b \,\&\, c) \vee (a \,\&\, \neg b \,\&\, c),$$

so its negation F takes the CNF form

$$(a \vee b \vee c) \,\&\, (a \vee \neg b \vee c) \,\&\, (a \vee \neg b \vee \neg c) \,\&\, (\neg a \vee b \vee \neg c).$$

This means that the formula $s_{1,1,3} = 1 \Leftrightarrow (s_{1,1,2} \vee \neg s_{3,2,2})$ takes the form

$$(s_{1,1,3} \vee s_{1,1,2} \vee s_{3,2,2}) \,\&\, (s_{1,1,3} \vee \neg s_{1,1,2} \vee s_{3,2,2}) \,\&$$

$$(s_{1,1,3} \vee \neg s_{1,1,2} \vee \neg s_{3,2,2}) \,\&\, (\neg s_{1,1,3} \vee s_{1,1,2} \vee \neg s_{3,2,2}).$$

The resulting long formula. The resulting formula should include:

- the CNF forms of all the formulas describing the state's dynamics,
- the fact that the initial value x is given; for example, for $x = 0$, it should be $s_{1,1,1} = 0$, i.e., $\neg s_{1,1,1}$; and
- the fact that the result of checking the property $C(x, y)$ is "true"; according to our computation scheme, this result is stored in the x-cell at moment 3, so this requirement takes the form $s_{1,1,3} = 1$, i.e., in CNF form, as $s_{1,1,3}$.

Thus, the corresponding long formula takes the following form:

$$\neg s_{3,1,1} \,\&\, \neg s_{3,2,1} \,\&$$

$$(s_{1,1,2} \vee \neg s_{1,1,1}) \,\&\, (\neg s_{1,1,2} \vee s_{1,1,1}) \,\&$$

$$\neg s_{2,1,2} \,\&\, s_{3,1,2} \,\&$$

$$(s_{3,2,2} \vee \neg s_{2,1,1}) \,\&\, (\neg s_{3,2,2} \vee s_{2,1,1}) \,\&$$

$$(s_{1,1,3} \vee s_{1,1,2} \vee s_{3,2,2}) \,\&\, (s_{1,1,3} \vee \neg s_{1,1,2} \vee s_{3,2,2}) \,\&$$

$$(s_{1,1,3} \vee \neg s_{1,1,2} \vee \neg s_{3,2,2}) \,\&\, (\neg s_{1,1,3} \vee s_{1,1,2} \vee \neg s_{3,2,2}) \,\&$$

$$\neg s_{2,1,3} \,\&\, \neg s_{3,1,3} \,\&\, s_{3,2,3} \,\&$$

$$\neg s_{1,1,1} \,\&\, s_{1,1,3}.$$

This formula says that for given $x = 0$ and for some y, we performed the checking of the property $C(x, y) \equiv (x \vee \neg y)$ and concluded that the result of checking is "true". Once the formula is satisfied, we can find y as the original value of the y-cell, i.e., as $y = s_{2,1,1}$.