

## Solution to Problem 40

**Homework 40.** Use the variable-elimination algorithm for checking satisfiability of 2-SAT formulas that we had in class to find the values that satisfy the following formula:

$$(\neg a \vee b) \& (a \vee b) \& (\neg a \vee \neg b) \& (\neg a \vee c) \& (a \vee c) \& (b \vee c).$$

**Solution.** In this formula, we have three Boolean variables:  $a$ ,  $b$ , and  $c$ . According to the algorithm, we need to eliminate them one by one.

Let us first the variable  $a$ . According to the general algorithm:

- each clause of the type  $a \vee x$  is converted to an inequality  $\neg x \leq a$ , and
- each clause of the type  $\neg a \vee c$  is converted into an inequality  $a \leq c$ .

Thus, in the above formula, clauses containing  $a$  or  $\neg a$  are converted into the following inequalities:

- the clause  $\neg a \vee b$  is converted into an inequality  $a \leq b$ ;
- the clause  $a \vee b$  is converted into an inequality  $\neg b \leq a$ ;
- the clause  $\neg a \vee \neg b$  is converted into an inequality  $a \leq \neg b$ ;
- the clause  $\neg a \vee c$  is converted into an inequality  $a \leq c$ ; and
- the clause  $a \vee c$  is converted into an inequality  $\neg c \leq a$ .

Here, we have:

- two lower bounds for  $a$ :  $\neg b \leq a$  and  $\neg c \leq a$ , and
- three upper bounds for  $a$ :  $a \leq b$ ,  $a \leq \neg b$ , and  $a \leq c$ .

In other words, we have:

$$\neg b, \neg c \leq a \leq b, \neg b, c. \tag{1}$$

Each lower bound must be smaller than or equal to each upper bound. So, we get the following inequalities:

- from the first lower bound, we get  $\neg b \leq b$ ,  $\neg b \leq \neg b$ , and  $\neg b \leq c$ ;

- from the second lower bound, we get  $\neg c \leq b$ ,  $\neg c \leq \neg b$ , and  $\neg c \leq c$

As in the example given in the handout, the inequality  $\neg b \leq b$  is only true when  $b = 1$ . Similarly, the inequality  $\neg c \leq c$  is only satisfied when  $c = 1$ . For these values  $b$  and  $c$ , the inequality (1) takes the form

$$0, 0 \leq a \leq 1, 0, 1,$$

i.e., the form  $0 \leq a \leq 0$ . Thus,  $a = 0$ .

So, the solution is:  $a = 0$ ,  $b = 1$ , and  $c = 1$ . In other words,  $a$  is false, and  $b$  and  $c$  are both true.