Solution to Problem 13

Problem. Similarly to a Turing machine that we had in class, that copies a number in unary code, design a Turing machine that copies a binary number. Test it on the example of a binary number 1101 (stored as 1011). The result should be 1011_1011, with a blank space in between.

Hint: instead of marking 1s, mark both 0s and 1s; instead of the state carry1in1st, we can now have two different states: carry0in1st and carry1in1st.

Solution. First, we start moving:

- start, – $\to$ R, in1st

If we see blank, this means that we had nothing to copy – the original number was 0. So, we go back and halt.

- in1st, – $\to$ L, halt

If we see 0 or 1, we mark it and go to the state carry0in1st or carry1in1st:

- in1st, 1 $\to \widehat{1}$, R, carry1in1st
- in1st, 0 $\to \widehat{0}$, R, carry0in1st

We move step by step inside the 1st (original) number:

- carry1in1st, 1 $\to$ R
- carry1in1st, 0 $\to$ R
- carry0in1st, 1 $\to$ R
- carry0in1st, 0 $\to$ R

Once we reach a blank space, we know that after moving one step to the right we will be in the 2nd number:

- carry1in1st, – $\to$ R, carry1in2nd
- carry0in1st, – $\to$ R, carry0in2nd

As long as we see 0s and 1s, we continue going through the 2nd number:

- carry1in2nd, 1 $\to$ R

- carry1in2nd, 0 → R
- carry0in2nd, 1 → R
- carry0in2nd, 0 → R

Once we see the first blank space, we drop the carried bit there and start going back:

- carry1in2nd, – → 1, L, backIn2nd
- carry0in2nd, – → 0, L, backIn2nd

We go left through the second number:

- backIn2nd, 1 → L
- backIn2nd, 0 → L

Once we read the blank space separating the second number from the first one, we need to check if we are done:

- backIn2nd, – → L, checkIfDone

If the symbol that we see in the 1st number is an unmarked symbol, this means that we are not done, so we need to go back and start finding the first unmarked symbols:

- checkIfDone, 1 → L, backIn1st
- checkIfDone, 0 → L, backIn1st

As we go left, if we see an unmarked symbol, we continue going left:

- backIn1st, 1 → L
- backIn1st, 0 → L

Once we meet a marked symbol, this means that next one to the right if the first unmarked on. So we go to the state in1st to repeat the whole procedure:

- backIn1st, $\widehat{1}$ → R, in1st
- backIn1st, $\widehat{0}$ → R, in1st

If the first symbol we see after getting into the 1st number is marked, this means that there are no more unmarked symbols in the 1st number. So, we unmark the marked symbol that we see and go to the unmark state:

- checkIfDone, $\widehat{1}$ → 1, L, unmark
- checkIfDone, $\widehat{0}$ → 0, L, unmark

Then, we go left step by step and unmark all the symbols of the first number one by one:

- unmark, $\widehat{1} \to 1$, L
- unmark, $\widehat{0} \to 0$, L

Once we reach the very first (blank) cell of the Turing machine, this means that we are done. So we halt:

- unmark, $- \to$ halt

Let us trace this Turing machine on the example of the word 1011.

$\underline{-}$	1	0	1	1	−	−	−	−	−	−	...	start
−	$\underline{1}$	0	1	1	−	−	−	−	−	−	...	in1st
−	$\widehat{1}$	$\underline{0}$	1	1	−	−	−	−	−	−	...	carry1in1st
−	$\widehat{1}$	$\underline{0}$	1	1	−	−	−	−	−	−	...	carry1in1st
−	$\widehat{1}$	0	$\underline{1}$	1	−	−	−	−	−	−	...	carry1in1st
−	$\widehat{1}$	0	1	$\underline{1}$	−	−	−	−	−	−	...	carry1in1st
−	$\widehat{1}$	0	1	1	$\underline{-}$	−	−	−	−	−	...	carry1in1st
−	$\widehat{1}$	0	1	1	−	$\underline{-}$	−	−	−	−	...	carry1in2nd
−	$\widehat{1}$	0	1	1	$\underline{-}$	1	−	−	−	−	...	backIn2nd
−	$\widehat{1}$	0	1	$\underline{1}$	−	1	−	−	−	−	...	checkIfDone
−	$\widehat{1}$	0	$\underline{1}$	1	−	1	−	−	−	−	...	backIn1st
−	$\widehat{1}$	$\underline{0}$	1	1	−	1	−	−	−	−	...	backIn1st
−	$\underline{\widehat{1}}$	0	1	1	−	1	−	−	−	−	...	backIn1st
−	$\widehat{1}$	$\underline{0}$	1	1	−	1	−	−	−	−	...	in1st
−	$\widehat{1}$	$\widehat{0}$	$\underline{1}$	1	−	1	−	−	−	−	...	carry0in1st
−	$\widehat{1}$	$\widehat{0}$	1	$\underline{1}$	−	1	−	−	−	−	...	carry0in1st
−	$\widehat{1}$	$\widehat{0}$	1	1	$\underline{-}$	1	−	−	−	−	...	carry0in1st
−	$\widehat{1}$	$\widehat{0}$	1	1	−	$\underline{1}$	−	−	−	−	...	carry0in2nd
−	$\widehat{1}$	$\widehat{0}$	1	1	−	1	$\underline{-}$	−	−	−	...	carry0in2nd
−	$\widehat{1}$	$\widehat{0}$	1	1	−	$\underline{1}$	0	−	−	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	1	1	$\underline{-}$	1	0	−	−	−	...	backIn2nd

−	$\widehat{1}$	$\widehat{0}$	1	$\underline{1}$	−	1	0	−	−	−	...	checkIfDone
−	$\widehat{1}$	$\widehat{0}$	$\underline{1}$	1	−	1	0	−	−	−	...	backIn1st
−	$\widehat{1}$	$\underline{\widehat{0}}$	1	1	−	1	0	−	−	−	...	backIn1st
−	$\widehat{1}$	$\widehat{0}$	$\underline{1}$	1	−	1	0	−	−	−	...	in1st
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\underline{1}$	−	1	0	−	−	−	...	carry1in1st
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	1	$\underline{-}$	1	0	−	−	−	...	carry1in1st
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	1	−	$\underline{1}$	0	−	−	−	...	carry1in2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	1	−	1	$\underline{0}$	−	−	−	...	carry1in2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	1	−	1	0	$\underline{-}$	−	−	...	carry1in2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	1	−	1	$\underline{0}$	1	−	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	1	−	$\underline{1}$	0	1	−	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	1	$\underline{-}$	1	0	1	−	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\underline{1}$	−	1	0	1	−	−	...	checkIfDone
−	$\widehat{1}$	$\widehat{0}$	$\underline{\widehat{1}}$	1	−	1	0	1	−	−	...	backIn1st
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\underline{1}$	−	1	0	1	−	−	...	in1st
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	$\underline{-}$	1	0	1	−	−	...	carry1in1st
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	−	$\underline{1}$	0	1	−	−	...	carry1in2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	−	1	$\underline{0}$	1	−	−	...	carry1in2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	−	1	0	$\underline{1}$	−	−	...	carry1in2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	−	1	0	1	$\underline{-}$	−	...	carry1in2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	−	1	0	$\underline{1}$	1	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	−	1	$\underline{0}$	1	1	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	−	$\underline{1}$	0	1	1	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\widehat{1}$	$\underline{-}$	1	0	1	1	−	...	backIn2nd
−	$\widehat{1}$	$\widehat{0}$	$\widehat{1}$	$\underline{\widehat{1}}$	−	1	0	1	1	−	...	checkIfDone
−	$\widehat{1}$	$\widehat{0}$	$\underline{\widehat{1}}$	1	−	1	0	1	1	−	...	unmark
−	$\widehat{1}$	$\underline{\widehat{0}}$	1	1	−	1	0	1	1	−	...	unmark

–	$\underline{\widehat{1}}$	0	1	1	–	1	0	1	1	–	...	unmark
$\underline{-}$	1	0	1	1	–	1	0	1	1	–	...	unmark
$\underline{-}$	1	0	1	1	–	1	0	1	1	–	...	halt