Solution to Problem 15

Problem. Sketch an example of a Turing machine for implementing mu-recursion, the way we did it in class, on the example of a function $\mu m.(m = a)$.

Solution. The given function is a particular case of a general μ-recursion expression

$$f(n_1, \ldots, n_k) = \mu m.P(n_1, \ldots, n_k, m)$$

corresponding to $k = 1$ and $P(n_1, m) \Leftrightarrow m = n_1$.

Suppose that we have a Turing machine for computing the equality relation $P(n_1, m)$. According to the general algorithm described in the lecture, we start with the state

$\underline{-}$	n_1	$-$	$\ldots$

start

and we want to end up in the state

$\underline{-}$	$f(n_1)$	$-$	$\ldots$

halt

Let us show how this can be done. First, we add 0 after the input, copy the whole tuple $(n_1, 0)$, and move the head before the second copy of n_1:

$\underline{-}$	n_1	$-$	0	$\underline{-}$	n_1	$-$	0	$-$	$\ldots$

Then, we apply the Turing machine computing the function $P(n_1, 0)$. As a result, we get the following state:

$-$	n_1	$-$	0	$\underline{-}$	$P(n_1, 0)$	$-$	$\ldots$

If $P(n_1, 0) = 0$ (i.e., if the property $P(n_1, m)$ is false), then we increase 0 by 1, copy the tuple $(n_1, 1)$:

$-$	n_1	$-$	1	$\underline{-}$	n_1	$-$	1	$-$	$\ldots$

and again apply the Turing machine for computing $P(n_1, m)$, resulting in:

$\underline{-}$	n_1	$-$	1	$\underline{-}$	$P(n_1, 1)$	$-$	$\ldots$

In general, at each iteration, we start with the state

$-$	n_1	$-$	m	$\underline{-}$	$P(n_1, m)$	$-$	$\ldots$

If $P(n_1, m) = 0$ (i.e., to "false"), then we increase m by 1, copy the tuple $(n_1, m+1)$:

$-$	n_1	$-$	$m+1$	$\underline{-}$	n_1	$-$	$m+1$	$-$	$\ldots$

and again apply the Turing machine for computing $P(n_1, m+1)$, resulting in:

$-$	n_1	$-$	$m+1$	$\underline{-}$	$P(n_1, m+1)$	$-$	$\ldots$

etc.

This continues until we get the first value m for which $P(n_1, m) = 1$ (i.e., "true"). In this case, we get the state

$-$	n_1	$-$	m	$\underline{-}$	1	$-$	$\ldots$

Here, the desired value m is 2-nd out of 3, so it can be found if we apply the corresponding projection π^3_2, resulting in:

$\underline{-}$	m	$-$	$\ldots$

halt

where $m = f(n_1) = \mu m.P(n_1, m)$.

This is exactly what we wanted.