

Solution to Problem 21

Problem. Let us consider cases when the set A is decidable and the sets B and C are r.e. Give four examples of such cases:

- an example when the union $A \cup B \cup C$ of the three sets is decidable,
- an example when the union $A \cup B \cup C$ of the three sets is not decidable,
- an example when the intersection of the three sets is decidable, and
- an example when the intersection of the three sets is not decidable.

Solution. Finding decidable examples is easy: it is sufficient to have decidable sets A , B , and C . For example:

- if $A = B = C = \emptyset$, then their union $A \cup B \cup C$ is the empty set and thus, decidable;
- if $A = B = C = \emptyset$, then their intersection $A \cap B \cap C$ is the empty set and thus, decidable.

Non-decidable examples are not so easy. To find such examples, let us recall that since the set A is decidable, it is also r.e. The union and intersection of r.e. sets are also r.e. The only r.e. set we know that is not decidable is the halting set H . So, to find examples of non-decidable, we need to find A , B , and C for which the corresponding union and intersection is H .

One of the three sets A , B , and C must be not decidable – otherwise, the union and intersection will be decidable. Since H is the only non-decidable set we know, one of the three sets must be equal to H . Here is one of the possible ways to find such examples:

- if $A = B = \emptyset$ and $C = H$, then their union $A \cup B \cup C$ is equal to H and is, thus, not decidable;
- if $A = B = N$ (the set of all natural numbers) and $C = H$, then their intersection $A \cap B \cap C$ is equal to H and is, thus, not decidable.