Solution to Problem 2

Problem 2. Prove that if $P(\overline{n})$, $Q(\overline{n})$, $f(\overline{n})$, $g(\overline{n})$, and $h(\overline{n})$ are all primitive recursive, then the function described as

$$F(\overline{n}) = \text{if } P(\overline{n}) \text{ then } f(\overline{n}) \text{ elseif } Q(\overline{n}) \text{ then } g(\overline{n}) \text{ else } h(\overline{n})$$

is also primitive recursive.

Solution. Proof. We want:

- the value $f(\overline{n})$ if $P(\overline{n})$ is true, i.e., if $P(\overline{n}) = 1$;
- the value $g(\overline{n})$ if $P(\overline{n})$ is false and $Q(\overline{n})$ is true, i.e., if the expression

  $$(\text{not } P(\overline{n})) \text{ and } Q(\overline{n})$$

  is true, i.e., equivalently, if $(1 \dot{-} P(\overline{n})) \cdot Q(\overline{n}) = 1$; and
- the value $h(\overline{n})$ if both $P(\overline{n})$ and $Q(\overline{n})$ are false, i.e., if the expression

  $$(\text{not } P(\overline{n})) \text{ and } (\text{not } Q(\overline{n}))$$

  is true, i.e., equivalently, if $(1 \dot{-} P(\overline{n})) \cdot (1 \dot{-} Q(\overline{n})) = 1$.

Thus, it makes sense to take

$$F(\overline{n}) = P(\overline{n}) \cdot f(\overline{n}) + (1 \dot{-} P(\overline{n})) \cdot Q(\overline{n}) \cdot g(\overline{n}) + (1 \dot{-} P(\overline{n})) \cdot (1 \dot{-} Q(\overline{n})) \cdot h(\overline{n}).$$

Then:

- if $P(\overline{n})$ is true, i.e., if $P(\overline{n}) = 1$, then $1 \dot{-} P(\overline{n}) = 0$, so

  $$F(\overline{n}) = 1 \cdot f(\overline{n}) + 0 \cdot g(\overline{n}) + 0 \cdot h(\overline{n}) = f(\overline{n});$$

- if $P(\overline{n}$ is false and $Q(\overline{n})$ is true, then $P(\overline{n}) = 0$ and $Q(\overline{n}) = 1$, then

  $$F(\overline{n}) = 0 \cdot f(\overline{n}) + 1 \cdot g(\overline{n}) + 0 \cdot h(\overline{n}) = g(\overline{n});$$

- if $P(\overline{n}$ and $Q(\overline{n})$ are both false, then $P(\overline{n}) = Q(\overline{n}) = 0$, then

  $$F(\overline{n}) = 0 \cdot f(\overline{n}) + 0 \cdot g(\overline{n}) + 1 \cdot h(\overline{n}) = h(\overline{n}).$$