

Solution to Problem 35

Problem. Use the variable-elimination algorithm for checking satisfiability of 2-SAT formulas that we had in class to find the values that satisfy the following formula:

$$(a \vee \neg b) \& (\neg a \vee b) \& (\neg a \vee \neg b) \& (\neg a \vee c) \& (a \vee c) \& (b \vee c).$$

Solution. In this formula, we have three Boolean variables: a , b , and c . According to the algorithm, we need to eliminate them one by one.

Let us first the variable a . According to the general algorithm:

- each clause of the type $a \vee x$ is converted to an inequality $\neg x \leq a$, and
- each clause of the type $\neg a \vee c$ is converted into an inequality $a \leq c$.

Thus, in the above formula, clauses containing a or $\neg a$ are converted into the following inequalities:

- the clause $a \vee \neg b$ is converted into an inequality $b \leq a$;
- the clause $\neg a \vee b$ is converted into an inequality $a \leq b$;
- the clause $\neg a \vee \neg b$ is converted into an inequality $a \leq \neg b$;
- the clause $\neg a \vee c$ is converted into an inequality $a \leq c$; and
- the clause $a \vee c$ is converted into an inequality $\neg c \leq a$.

Here, we have:

- two lower bounds for a : $b \leq a$ and $\neg c \leq a$, and
- three upper bounds for a : $a \leq b$, $a \leq \neg b$, and $a \leq c$.

In other words, we have:

$$b, \neg c \leq a \leq b, \neg b, c. \tag{1}$$

Each lower bound must be smaller than or equal to each upper bound. So, we get the following inequalities:

- from the first lower bound, we get $b \leq b$, $b \leq \neg b$, and $b \leq c$;

- from the second lower bound, we get $\neg c \leq b$, $\neg c \leq \neg b$, and $\neg c \leq c$

As in the example given in the lecture, the inequality $b \leq \neg b$ is only true when $b = 0$. Similarly, the inequality $\neg c \leq c$ is only satisfied when $c = 1$. For these values b and c , the inequality (1) takes the form

$$0, 0 \leq a \leq 0, 1, 1,$$

i.e., the form $0 \leq a \leq 0$. Thus, $a = 0$.

So, the solution is: $a = 0$, $b = 0$, and $c = 1$. In other words, a and b are false, and c is true.