

Solution to Homework 36

Problem. On the example of function $f(x) = 1$, trace, step by step, how Deutsch-Josza algorithm will conclude that $f(0) = f(1)$ while applying f only once.

Solution. The Deutsch-Josza algorithm consists of the following steps:

- we start with the state $|0, 1\rangle = |0\rangle \otimes |1\rangle$;
- we apply the Hadamard transformation H to both bits, i.e., the transformation for which

$$H(|0\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle; \quad H(|1\rangle) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle;$$

- then, we apply the function f , i.e., apply the transformation

$$f(|x, y\rangle) = |x, y \oplus f(x)\rangle,$$

where $a \oplus b$ means addition modulo 2 or, equivalently, exclusive “or”:

$$0 \oplus 0 = 0; \quad 0 \oplus 1 = 1 \oplus 0 = 1; \quad 1 \oplus 1 = 0;$$

- after that, we again apply the Hadamard transformation to both bits;
- finally, we measure the first bit of the resulting 2-bit state:
 - if the first bit is 0, we conclude that the function f is constant;
 - if the first bit is 1, we conclude that the function f is not constant.

According to the lecture, after applying the Hadamard transformation H to both bits of the state $|0, 1\rangle = |0\rangle \otimes |1\rangle$, we get the state

$$H(|0\rangle) \otimes H(|1\rangle) = \frac{1}{2}|0, 0\rangle - \frac{1}{2}|0, 1\rangle + \frac{1}{2}|1, 0\rangle - \frac{1}{2}|1, 1\rangle. \quad (1)$$

When we apply the function f , we get the following:

$$f(|0, 0\rangle) = |0, 1\rangle, \quad f(|0, 1\rangle) = |0, 0\rangle, \quad f(|1, 0\rangle) = |1, 1\rangle, \quad f(|1, 1\rangle) = |1, 0\rangle.$$

Thus, the state (1) gets transformed into

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{2}|0, 1\rangle - \frac{1}{2}|0, 0\rangle + \frac{1}{2}|1, 1\rangle - \frac{1}{2}|1, 0\rangle =$$

$$-\frac{1}{2}|0\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle \otimes |1\rangle - \frac{1}{2}|1\rangle \otimes |0\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle.$$

The first two terms have a common factor $|0\rangle$, the third and the fourth one have a common factor $|1\rangle$, so we have

$$f(H(|0\rangle) \otimes H(|1\rangle)) = -\frac{1}{\sqrt{2}}|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right) - \frac{1}{\sqrt{2}}|1\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right).$$

This expression can be equivalently reformulated as

$$f(H(|0\rangle) \otimes H(|1\rangle)) = -\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right).$$

For the first bit, we have $H|0\rangle$, for the second bit, we have is $H|1\rangle$:

$$f(H(|0\rangle) \otimes H(|1\rangle)) = -H(|0\rangle) \otimes H(|1\rangle).$$

It is known that when we apply the Hadamard transformation twice, we get back the original state. In particular, $H(H(|1\rangle)) = |1\rangle$. Thus, when we apply the Hadamard transformation to both bits once again, we get the state

$$-|0\rangle \otimes |1\rangle.$$

Measuring the value of the first bit, we get the value 0 with probability $|-1|^2 = 1$. Thus, we can indeed conclude that $f(0) = f(1)$ – and we called the function f only once.