Solution to Homework 37

Problem. What class of polynomial hierarchy contains $\Sigma_4\mathrm{P}^{\Pi_1\mathrm{P}}$? Explain your answer.

Solution. For each oracle A, the class $\Sigma_4\mathrm{P}^A$ is described by formulas F with 4 quantifiers starting with the existential quantifier in which the main property is feasible with respect to A:

$$F \equiv \exists x_1\, \forall x_2\, \exists x_3\, \forall x_4\, C^A(x_1, x_2, x_3, x_4, x). \tag{1}$$

Here, the fact that $C^{\Pi_1\mathrm{P}}$ is feasible with respect to the corresponding oracle $A = \Pi_1\mathrm{P}$ means that this property, in its turn, has the form

$$C^A(x_1, x_2, x_3, x_4, x) = C^{\Pi_1\mathrm{P}}(x_1, x_2, x_3, x_4, x) \equiv$$

$$\forall x_5\, C(x_1, x_2, x_3, x_4, x_5, x), \tag{2}$$

where the property $C(x_1, x_2, x, x_3, x_4, x_5, x)$ is actually feasible.

Substituting the formula (2) into the expression (1), we get

$$F \equiv \exists x_1\, \forall x_2\, \exists x_3\, \forall x_4\, \forall x_5\, C(x_1, x_2, x_3, x_4, x_5, x). \tag{3}$$

As mentioned in the lecture, two consequent quantifiers – such as $\forall x_4\, \forall x_5$ – is equivalent to $\forall(x_4, x_5)$, and can thus be compressed into a single quantifier $\forall x'_4$ – since, as we have learned earlier, there is a feasible correspondence between natural numbers and pairs of natural numbers. In this correspondence, each code n of the pair can be transformed back into the pair $(\pi_1(n), \pi_2(n))$.

Thus, the formula (3) takes the simplified form

$$F \equiv \exists x_1\, \forall x_2\, \exists x_3\, \forall x'_4\, C(x_1, x_2, x_3, \pi_1(x'_4), \pi_2(x'_4), x).$$

In this expression, we have 4 quantifiers, the first of which is the existential quantifier. Thus, this formula belongs to the class $\Sigma_4\mathrm{P}$, i.e.:

$$\Sigma_4\mathrm{P}^{\Pi_1\mathrm{P}} \subseteq \Sigma_4\mathrm{P}.$$