Solution to Problem 3

Problem. Prove, from scratch – i.e., using only the definition of the primitive recursive function (and not using any results that we had in class without proving them) – that the function $a \,/\, b + a \,\%\, b$ is primitive recursive.

Solution. To prove that the desired function is primitive recursive (p.r.), we will show:

- that addition is p.r.,
- that remainder is p.r.. and
- that division is p.r.

1. Addition is p.r. since the function

$$add(a,b) = a + b = a + 1 + \ldots + 1 \text{ (}b\text{ times)}$$

can be represented as

$$add(a,0) = a;$$

$$add(a,b+1) = add(a,b) + 1.$$

2. The remainder function $rem(a,b) = b \,\%\, a$ can be represented as follows:

$$rem(a,0) = 0;$$

$$rem(a,b+1) = if\ (rem(a,b) + 1 < b)\ then\ (rem(a,b) + 1)\ else\ 0.$$

To show that this expression is p.r., we need to show:

- that $<$ is p.r., and
- that the if-then-else construction is p.r.

2.1. Let us first show that the relation $r < b$ is p.r. Indeed, the condition $r < b$ is equivalent to $b \dot{-} r > 0$, where:

- $b \dot{-} r = b - r$ if $b > r$ and
- $b \dot{-} r = 0$ otherwise.

So, to prove that $r < b$ is p.r., it is sufficient to prove that the relation $a > 0$ and subtraction $sub(b, r) = b \dot{-} r$ are p.r.

2.1.1. To prove that $a > 0$ is p.r., we can use the following description of this relation:

$$pos(0) = 0;$$
$$pos(m + 1) = 1.$$

2.1.2. Subtraction can be represented as

$$sub(b, 0) = b;$$
$$sub(b, r + 1) = sub(b, r) \dot{-} 1.$$

So, to show that subtraction is p.r., it is sufficient to prove the function $prev(n) = n \dot{-} 1$ is p.r.

2.1.3. Indeed, the function $prev(n)$ can be represented as follows:

$$prev(0) = 0;$$
$$prev(n + 1) = n.$$

2.2. Let us no show that the if-the-else construction is p.r. Indeed, the if-then-else construction can be represented as

$$if\ (P(\overline{n}))\ then\ f(\overline{n})\ else\ g(\overline{n}) = P(\overline{n}) \cdot f(\overline{n}) + (1 \dot{-} P(\overline{n})) \cdot g(\overline{n}).$$

So, to complete the proof that remainder is p.r., it is sufficient to prove that multiplication is primitive recursive.

2.2.1 Indeed, multiplication can be represented as

$$mult(a, b) = a \cdot b = a + \ldots + a \text{ (b times)},$$

thus

$$mult(a, 0) = 0;$$
$$mult(a, b + a) = mult(a, b) + a.$$

So, remainder is p.r.

3. Let us now show that division is p.r. Indeed, division $div(a, b) = b\,/\,a$ can be represented as follows:

$$div(a, 0) = 0;$$
$$div(a, b + 1) = if\ (rem(a, b + 1) > 0)\ then\ div(a, b)\ else\ (div(a, b) + 1).$$

Here, if-then-else construction, remainder, and $>$ are all p.r., so division is also p.r.

4. Since addition, remainder, and division are all p.r., the desired function – which is their composition – is also p.r.