

Solution to Problem 12

Problem. In class, we described a Turing machine that computes $g(n) = n + 1$. In Homework 9, you designed a Turing machine that computes a function $f(n)$ which is equal to $n - 3$ when $n \geq 3$ and to 0 when $n = 0$.

In class, we described the general algorithm for designing a Turing machine that computes the composition of two functions. The assignment is to use this general algorithm to design a Turing machine that computes the composition $g(f(n))$. Trace, step-by-step, on an example, how your Turing machine works. For example, you can take as input $n = 2$.

Reminder: The Turing machine for computing $g(n) = n + 1$ for a unary input n is based on the following idea:

- we go step-by-step until we find the first blank space,
- then, we replace this blank space with 1 and go back.

This machine has the following rules:

- start, $- \rightarrow R$, working
(we start going to the right)
- working, $1 \rightarrow R$
(we see 1, so we continue going),
- working, $- \rightarrow 1, L$, back
(we see a blank space, so we replace it with 1 and start going back)
- back, $\rightarrow L$
(while we see 1s, we continue going back)
- back, $- \rightarrow \text{halt}$
(once we reach the very first cell, we stop).

The Turing machine for computing $f(n)$ is based on the following idea:

- we go step-by-step until we find the first blank space,
- then, we go back, replace the last two 1s with blanks, and go back all the way.

We need to take special care of the case when $n = 0$.

Solution. The resulting Turing machine takes the following form:

- start, $- \rightarrow R$, moving₁
- moving₁, $1 \rightarrow R$
- moving₁, $- \rightarrow$ erasing1st₁
- erasing1st₁, $1 \rightarrow -, L$, erasing2nd₁
- erasing2nd₁, $1 \rightarrow -, L$, erasing3rd₁
- erasing3rd₁, $1 \rightarrow -, L$, back₁
- back₁, $1 \rightarrow L$
- back₁, $- \rightarrow$ start₂
- erasing1st₁, $- \rightarrow$ start₂
- erasing2nd₁, $- \rightarrow$ start₂
- erasing3rd₁, $- \rightarrow$ start₂
- start₂, $- \rightarrow R$, working₂
- working₂, $1 \rightarrow R$
- working₂, $- \rightarrow 1, L$, back₂
- back₂, $\rightarrow L$
- back₂, $- \rightarrow$ halt

Let us trace it for $n = 2$;

-	1	1	-	...	start
-	<u>1</u>	1	-	...	moving ₁
-	1	<u>1</u>	-	...	moving ₁
-	1	1	<u>-</u>	...	moving ₁
-	1	<u>1</u>	-	...	erasing1st ₁
-	<u>1</u>	-	-	...	erasing2nd ₁
<u>-</u>	-	-	-	...	erasing3rd ₁
<u>-</u>	-	-	-	...	start ₂
-	<u>-</u>	-	-	...	working ₂
<u>-</u>	1	-	-	...	back ₂
<u>-</u>	1	-	-	...	halt