Solution to Problem 14

Problem. Sketch an example of a Turing machine for implementing primitive recursion (i.e., a for-loop), the way we did it in class, on the example of the following simple for-loop

```
sum = a;
for(int i = 1; i <= b; i++)
  {sum = sum + c;}
```

No details are required, but any details will give you extra credit.

Solution. In mathematical terms, the above for-loop takes the following form:

$$sum(a, 0) = a;$$

$$sum(a, m+1) = sum(a, m) + c.$$

After we rename the function sum into h and the parameters a and c into n_1 and n_2, we get the standard form:

$$h(n_1, n_2, 0) = n_1;$$

$$h(n_1, n_2, m+1) = h(n_1, n_2, m) + n_2.$$

In this standard form, we have $f(n_1, n_2) = n_1$, i.e., $f = \pi^2_1$, and $g(n_1, n_2, m, h) = h + n_2$, i.e., $g = add(\pi^4_4, \pi^4_2)$.

Let us follow the general scheme for computing primitive recursion. Suppose that we have Turing machines computing the functions $f(n_1, n_2) = n_1$ and $g(n_1, n_2, m, h) = h + n_2$. Let us show how to build a Turing machine that compute the desired function $h = PR(f, g)$. We start with the state

$\underline{-}$	n_1	$-$	n_2	$-$	x	$-$	$\ldots$	start

and we want to end up in the state

$\underline{-}$	$h(n_1, n_2, x)$	$-$	$\ldots$	halt

This can be done as follows. First, we copy x, add 0, then copy the numbers n_1 and n_2, and move the head into the cell right before the second copy of n_1:

$-$	n_1	$-$	n_2	$-$	x	$-$	x	$-$	0	$\underline{-}$	n_1	$-$	n_2	$-$	$\ldots$

Then, we apply the Turing machine f. Since a Turing machine never goes beyond the cell where it starts, it will compute the value

$$h(n_1, n_2, 0) = f(n_1, n_2) = n_1,$$

so we will have the following state of the tape:

$-$	n_1	$-$	n_2	$-$	x	$-$	x	$-$	0	$\underline{-}$	$h(n_1, n_2, 0)$	$-$	$\ldots$

Now, we copy n_1, n_2, and 0 before h, and get

$-$	n_1	$-$	n_2	$-$	x	$-$	x	$-$	0	$\underline{-}$	n_1	$-$	n_2	$-$	0	$-$	$h(n_1, n_2, 0)$	$-$	$\ldots$

Then, we apply the Turing machine for computing the function g, and get $h(n_1, n_2, 1) = g(n_1, n_2, 0, h(n_1, n_2, 0))$. So, the tape has the form:

$-$	n_1	$-$	n_2	$-$	x	$-$	x	$-$	0	$\underline{-}$	$h(n_1, n_2, 1)$	$-$	$\ldots$

After that, we decrease the second copy of x by 1, increase 0 by 1, and get the following:

$-$	n_1	$-$	n_2	$-$	x	$-$	$x-1$	$-$	1	$\underline{-}$	$h(n_1, n_2, 1)$	$-$	$\ldots$

and we repeat a similar procedure.

In general, for each $m \leq x$, we get the following state of the tape:

$-$	n_1	$-$	n_2	$-$	x	$-$	$x-m$	$-$	m	$\underline{-}$	$h(n_1, n_2, m)$	$-$	$\ldots$

Then, we copy n_1, n_2, and m before h, and get

$-$	n_1	$-$	n_2	$-$	x	$-$	$x-m$	$-$	m	$\underline{-}$	n_1	$-$	n_2	$-$	m	$-$	$h(n_1, n_2, m)$	$-$	$\ldots$

Now, we apply the Turing machine for computing the function g, and get

$$h(n_1, n_2, m+1) = h(n_1, n_2, m, h(n_1, n_2, m)).$$

So, the tape has the form:

$-$	n_1	$-$	n_2	$-$	x	$-$	$x-m$	$-$	m	$\underline{-}$	$h(n_1, n_2, m+1)$	$-$	$\ldots$

Then, we check whether $x - m = 0$. If $x - m > 0$, we decrease $x - m$ by 1, increase m by 1, and get the following:

$-$	n_1	$-$	n_2	$-$	x	$-$	$x-(m+1)$	$-$	$m+1$	$\underline{-}$	$h(n_1, n_2, m+1)$	$-$	$\ldots$

and we repeat a similar procedure.

Once we get $x - m = 0$ i.e., $m = x$, the state of the tape takes the form

$-$	n_1	$-$	n_2	$-$	x	$-$	0	$-$	x	$\underline{-}$	$h(n_1, n_2, x)$	$-$	$\ldots$

Here, we have $k + 4 = 6$ numbers:

- the numbers n_1 and n_2, and
- four numbers x, 0, x, and $h(n_1, n_2, x)$.

The desired value $h(n_1, n_2, x)$ is 6-th out of 6, so we can get it by applying the Turing machine computing the corresponding projection π_6^6:

$\underline{-}$	$h(n_1, n_2, x)$	$-$	$\ldots$	halt

This is exactly what we wanted.

In this construction, we use composition, adding 1, subtracting 1, copying, and projection. We know how to do all this on a Turing machine, so indeed we can thus build a Turing machine for computing the function $PR(f, g)$.