Solution to Problem 15

Problem. Sketch an example of a Turing machine for implementing µ-recursion, the way we did it in class, on the example of a function \( \mu m.(m > a) \).

Solution. The given function is a particular case of a general µ-recursion expression
\[
f(n_1, \ldots, n_k) = \mu m.P(n_1, \ldots, n_k, m)
\]
corresponding to \( k = 1 \) and \( P(n_1, m) \Leftrightarrow n_1 > m \).

Suppose that we have a Turing machine for computing the inequality relation \( P(n_1, m) \). According to the general algorithm described in the lecture, we start with the state
\[
\text{– } n_1 \text{ – } \ldots \text{ start}
\]
and we want to end up in the state
\[
\text{– } f(n_1) \text{ – } \ldots \text{ halt}
\]

Let us show how this can be done. First, we add 0 after the input, copy the whole tuple \((n_1, 0)\), and move the head before the second copy of \(n_1\):
\[
\text{– } n_1 \text{ – } 0 \text{ – } n_1 \text{ – } 0 \text{ – } \ldots
\]

Then, we apply the Turing machine computing the function \( P(n_1, 0) \). As a result, we get the following state:
\[
\text{– } n_1 \text{ – } 0 \text{ – } P(n_1, 0) \text{ – } \ldots
\]

If \( P(n_1, 0) = 0 \) (i.e., if the property \( P(n_1, m) \) is false), then we increase 0 by 1, copy the tuple \((n_1, 1)\):
\[
\text{– } n_1 \text{ – } 1 \text{ – } n_1 \text{ – } 1 \text{ – } \ldots
\]

and again apply the Turing machine for computing \( P(n_1, m) \), resulting in:
\[
\text{– } n_1 \text{ – } 1 \text{ – } P(n_1, 1) \text{ – } \ldots
\]

In general, at each iteration, we start with the state
\[
\text{– } n_1 \text{ – } m \text{ – } P(n_1, m) \text{ – } \ldots
\]

If \( P(n_1, m) = 0 \) (i.e., to “false”), then we increase \( m \) by 1, copy the tuple \((n_1, m + 1)\):
\[
\text{– } n_1 \text{ – } m + 1 \text{ – } n_1 \text{ – } m + 1 \text{ – } \ldots
\]
and again apply the Turing machine for computing $P(n_1, m + 1)$, resulting in:

\[ n_1 \quad m + 1 \quad P(n_1, m + 1) \quad \ldots \]

etc.

This continues until we get the first value $m$ for which $P(n_1, m) = 1$ (i.e., “true”). In this case, we get the state

\[ n_1 \quad m \quad 1 \quad \ldots \]

Here, the desired value $m$ is 2-nd out of 3, so it can be found if we apply the corresponding projection $\pi_3^2$, resulting in:

\[ m \quad \ldots \quad \text{halt} \]

where $m = f(n_1) = \mu m. P(n_1, m)$.

This is exactly what we wanted.