Solution to Problem 1

**Problem 1.** Prove that the function computing the sum $2+3+\ldots+n+(n+1)$ is primitive recursive. This proof should follow the same pattern that we used in class to prove that addition and multiplication are primitive recursive:

- You start with a 3-dot expression.
- First you write a for-loop corresponding to this function.
- Then you describe this for-loop in mathematical terms.
- Then, to prepare for a match with the general expression for primitive recursion, you rename the function to $f$ and the parameters to $n_1, \ldots, n_k, m$.
- Then you write down the general expression of primitive recursion for the corresponding $k$.
- Then you match: find $f$ and $g$ for which the specific case of primitive recursion will be exactly the functions corresponding to initialization and to what is happening inside the loop.
- Then, you get a final expression for the function $2+3+\ldots+n+(n+1)$ that proves that this function is primitive recursive, i.e., that it can be formed from $0$, $\pi^k_i$, and $\sigma$ by composition and primitive recursion.

**Solution.** Here is the for-loop for computing the desired expression:

```c
int sum = 0;
for (int i = 1; i <= m; i++){
    sum = c + (i + 1);}
```

Let us now describe this for-loop in mathematical terms. After iteration $i$, we add the number $i+1$. The value $sum(i)$ is the value of the variable $sum$ after iteration $i$. So, $sum(m+1)$ is the value of the variable $sum$ after iteration $m+1$. In this case, $i = m + 1$, so we get:

$$sum(0) = 0;$$

$$sum(m + 1) = sum(m) + ((m + 1) + 1).$$
To prepare for the match, we rename the function to $h$ (here, there are no other parameters to rename):

\[
\begin{align*}
h(0) &= 0; \\
h(m + 1) &= h(m) + (m + 2).
\end{align*}
\]

Here, we are defining a function of 1 variable. In general, primitive recursion defines a function of $k + 1$ variables. Here, $k + 1 = 1$, so $k = 0$, and the general expression for primitive recursion takes the following form:

\[
\begin{align*}
h(0) &= f; \\
h(m + 1) &= g(m, h(m)).
\end{align*}
\]

To match with the above description, we need to take $f = 0$ and $g(m, h) = h + (m + 2)$, i.e., $g = \text{add}(\pi^2_2, \sigma \circ \sigma \circ \pi_1^2)$.

Thus, the desired expression for our function is

\[
\text{sum} = PR(0, \text{add}(\pi^2_2, \sigma \circ \sigma \circ \pi_1^2))
\]