Solution to Problem 21

Problem. Let us consider cases when the sets A and B are decidable and the set C is r.e. Give four examples of such cases:

- an example when the union $A \cup B \cup C$ of the three sets is decidable,
- an example when the union $A \cup B \cup C$ of the three sets is not decidable,
- an example when the intersection of the three sets is decidable, and
- an example when the intersection of the three sets is not decidable.

Solution. Finding decidable examples is easy: it is sufficient to have decidable sets A, B, and C. For example:

- if $A = B = C = \emptyset$, then their union $A \cup B \cup C$ is the empty set and thus, decidable;
- if $A = B = C = \emptyset$, then their intersection $A \cap B \cap C$ is the empty set and thus, decidable.

Non-decidable examples are not so easy. To find such examples, let us recall that since the sets A and B are decidable, they are also r.e. The union and intersection of r.e. sets are also r.e. The only r.e. set we know that is not decidable is the halting set H. So, to find examples of non-decidable, we need to find A, B, and C for which the corresponding union and intersection is H.

One of the three sets A, B, and C must be not decidable – otherwise, the union and intersection will be decidable. Since H is the only non-decidable set we know, one of the three sets must be equal to H. Here is one of the possible ways to find such examples:

- if $A = B = \emptyset$ and $C = H$, then their union $A \cup B \cup C$ is equal to H and is, thus, not decidable;
- if $A = B = N$ (the set of all natural numbers) and $C = H$, then their intersection $A \cap B \cap C$ is equal to H and is, thus, not decidable.