

## Solution to Problem 2

**Problem 2.** Prove that if  $P(\bar{n})$ ,  $Q(\bar{n})$ ,  $R(\bar{n})$ ,  $f(\bar{n})$ ,  $g(\bar{n})$ ,  $h(\bar{n})$ , and  $i(\bar{n})$  are all primitive recursive, then the function described as

$$F(\bar{n}) = \text{if } P(\bar{n}) \text{ then } f(\bar{n}) \text{ elseif } Q(\bar{n}) \text{ then } g(\bar{n}) \text{ elseif } R(\bar{n}) \text{ then } h(\bar{n}) \text{ else } i(\bar{n})$$

is also primitive recursive.

**Solution. Proof.** We want:

- the value  $f(\bar{n})$  if  $P(\bar{n})$  is true, i.e., if  $P(\bar{n}) = 1$ ;
- the value  $g(\bar{n})$  if  $P(\bar{n})$  is false and  $Q(\bar{n})$  is true, i.e., if the expression

$$(\text{not } P(\bar{n})) \text{ and } Q(\bar{n})$$

is true, i.e., equivalently, if  $(1 \div P(\bar{n})) \cdot Q(\bar{n}) = 1$ ;

- the value  $h(\bar{n})$  if  $P(\bar{n})$  and  $Q(\bar{n})$  are false and  $R(\bar{n})$  is true, i.e., if the expression

$$(\text{not } P(\bar{n})) \text{ and } (\text{not } Q(\bar{n})) \text{ and } R(\bar{n})$$

is true, i.e., equivalently, if  $(1 \div P(\bar{n})) \cdot (1 \div Q(\bar{n})) \cdot R(\bar{n}) = 1$ ; and

- the value  $i(\bar{n})$  if all three statements  $P(\bar{n})$ ,  $Q(\bar{n})$  and  $R(\bar{n})$  are false, i.e., if the expression

$$(\text{not } P(\bar{n})) \text{ and } (\text{not } Q(\bar{n})) \text{ and } (\text{not } R(\bar{n}))$$

is true, i.e., equivalently, if  $(1 \div P(\bar{n})) \cdot (1 \div Q(\bar{n})) \cdot (1 \div R(\bar{n})) = 1$ .

Thus, it makes sense to take

$$F(\bar{n}) = P(\bar{n}) \cdot f(\bar{n}) + (1 \div P(\bar{n})) \cdot Q(\bar{n}) \cdot g(\bar{n}) + (1 \div P(\bar{n})) \cdot (1 \div Q(\bar{n})) \cdot R(\bar{n}) \cdot h(\bar{n}) + (1 \div P(\bar{n})) \cdot (1 \div Q(\bar{n})) \cdot (1 \div R(\bar{n})) \cdot i(\bar{n}).$$

Then:

- if  $P(\bar{n})$  is true, i.e., if  $P(\bar{n}) = 1$ , then  $1 \div P(\bar{n}) = 0$ , so

$$F(\bar{n}) = 1 \cdot f(\bar{n}) + 0 \cdot g(\bar{n}) + 0 \cdot h(\bar{n}) + 0 \cdot i(\bar{n}) = f(\bar{n});$$

- if  $P(\bar{n})$  is false and  $Q(\bar{n})$  is true, then  $P(\bar{n}) = 0$  and  $Q(\bar{n}) = 1$ , then

$$F(\bar{n}) = 0 \cdot f(\bar{n}) + 1 \cdot g(\bar{n}) + 0 \cdot h(\bar{n}) + 0 \cdot i(\bar{n}) = g(\bar{n});$$

- if  $P(\bar{n})$  and  $Q(\bar{n})$  are both false and  $R(\bar{n})$  is true, then  $P(\bar{n}) = Q(\bar{n}) = 0$  and  $R(\bar{n}) = 1$ , then

$$F(\bar{n}) = 0 \cdot f(\bar{n}) + 0 \cdot g(\bar{n}) + 1 \cdot h(\bar{n}) + 0 \cdot i(\bar{n}) = h(\bar{n});$$

- if all three statements  $P(\bar{n})$ ,  $Q(\bar{n})$ , and  $R(\bar{n})$  are false, then  $P(\bar{n}) = Q(\bar{n}) = R(\bar{n}) = 0$ , then

$$F(\bar{n}) = 0 \cdot f(\bar{n}) + 0 \cdot g(\bar{n}) + 0 \cdot h(\bar{n}) + 1 \cdot i(\bar{n}) = i(\bar{n}).$$