Solution to Homework 36

Problem. On the example of function $f(x) = x$, trace, step by step, how Deutsch-Josza algorithm will conclude that $f(0) \neq f(1)$ while applying $f$ only once.

Solution. The Deutsch-Josza algorithm consists of the following steps:

1. we start with the state $|0, 1⟩ = |0⟩ \otimes |1⟩$;
2. we apply the Hadamard transformation $H$ to both bits, i.e., the transformation for which
   
   $H(|0⟩) = \frac{1}{\sqrt{2}}|0⟩ + \frac{1}{\sqrt{2}}|1⟩; \quad H(|1⟩) = \frac{1}{\sqrt{2}}|0⟩ - \frac{1}{\sqrt{2}}|1⟩$;
3. then, we apply the function $f$, i.e., apply the transformation
   
   $f(|x, y⟩) = |x, y \oplus f(x)⟩$,
   
   where $a \oplus b$ means addition modulo 2 or, equivalently, exclusive “or”:
   
   $0 \oplus 0 = 0; \quad 0 \oplus 1 = 1; \quad 1 \oplus 0 = 1; \quad 1 \oplus 1 = 0$;
4. after that, we again apply the Hadamard transformation to both bits;
5. finally, we measure the first bit of the resulting 2-bit state:
   - if the first bit is 0, we conclude that the function $f$ is constant;
   - if the first bit is 1, we conclude that the function $f$ is not constant.

According to the lecture, after applying the Hadamard transformation $H$ to both bits of the state $|0, 1⟩ = |0⟩ \otimes |1⟩$, we get the state

$$H(|0⟩) \otimes H(|1⟩) = \frac{1}{2}|0, 0⟩ - \frac{1}{2}|0, 1⟩ + \frac{1}{2}|1, 0⟩ - \frac{1}{2}|1, 1⟩. \quad (1)$$

When we apply the function $f$, we get the following:

$$f(|0, 0⟩) = |0, 0⟩, \quad f(|0, 1⟩) = |0, 1⟩, \quad f(|1, 0⟩) = |1, 1⟩, \quad f(|1, 1⟩) = |1, 0⟩.$$ 

Thus, the state (1) gets transformed into

$$f(H(|0⟩) \otimes H(|1⟩)) = \frac{1}{2}|0, 0⟩ - \frac{1}{2}|0, 1⟩ + \frac{1}{2}|1, 1⟩ - \frac{1}{2}|1, 0⟩ = \ldots$$
\[
\frac{1}{2} |0\rangle \otimes |0\rangle - \frac{1}{2} |0\rangle \otimes |1\rangle - \frac{1}{2} |1\rangle \otimes |0\rangle + \frac{1}{2} |1\rangle \otimes |1\rangle.
\]

The first two terms have a common factor $|0\rangle$, the third and the fourth one have a common factor $|1\rangle$, so we have
\[
f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{\sqrt{2}} |0\rangle \otimes \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) - \frac{1}{\sqrt{2}} |1\rangle \otimes \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right).
\]

This expression can be equivalently reformulated as
\[
f(H(|0\rangle) \otimes H(|1\rangle)) = \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) \otimes \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right).
\]

For the first bit, we have $H|1\rangle$, for the second bit, we also have $H|1\rangle$:
\[
f(H(|0\rangle) \otimes H(|1\rangle)) = H(|1\rangle) \otimes H(|1\rangle).
\]

It is known that when we apply the Hadamard transformation twice, we get the same stat. In particular, $H(H(|1\rangle)) = |1\rangle$. Thus, when we apply the Hadamard transformation to both bits once again, we get the state
\[
|1\rangle \otimes |1\rangle.
\]

Measuring the value of the first bit, we get the value 1 with probability $|1\rangle^2 = 1$. Thus, we can indeed conclude that $f(0) \neq f(1)$ – and we called the function $f$ only once.