Solution to Homework 36

Problem. On the example of function $f(x) = x$, trace, step by step, how Deutsch-Josza algorithm will conclude that $f(0) \neq f(1)$ while applying f only once.

Solution. The Deutsch-Josza algorithm consists of the following steps:

- we start with the state $|0,1\rangle = |0\rangle \otimes |1\rangle$;
- we apply the Hadamard transformation H to both bits, i.e., the transformation for which
$$H(|0\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle; \quad H(|1\rangle) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle;$$
- then, we apply the function f, i.e., apply the transformation
$$f(|x,y\rangle) = |x, y \oplus f(x)\rangle,$$
where $a \oplus b$ means addition modulo 2 or, equivalently, exclusive "or":
$$0 \oplus 0 = 0; \quad 0 \oplus 1 = 1 \oplus 0 = 1; \quad 1 \oplus 1 = 0;$$
- after that, we again apply the Hadamard transformation to both bits;
- finally, we measure the first bit of the resulting 2-bit state:
  - if the first bit is 0, we conclude that the function f is constant;
  - if the first bit is 1, we conclude that the function f is not constant.

According to the lecture, after applying the Hadamard transformation H to both bits of the state $|0,1\rangle = |0\rangle \otimes |1\rangle$, we get the state

$$H(|0\rangle) \otimes H(|1\rangle) = \frac{1}{2}|0,0\rangle - \frac{1}{2}|0,1\rangle + \frac{1}{2}|1,0\rangle - \frac{1}{2}|1,1\rangle. \tag{1}$$

When we apply the function f, we get the following:

$$f(|0,0\rangle) = |0,0\rangle, \quad f(|0,1\rangle) = |0,1\rangle, \quad f(|1,0\rangle) = |1,1\rangle, \quad f(|1,1\rangle) = |1,0\rangle.$$

Thus, the state (1) gets transformed into

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{2}|0,0\rangle - \frac{1}{2}|0,1\rangle + \frac{1}{2}|1,1\rangle - \frac{1}{2}|1,0\rangle =$$

$$\frac{1}{2}|0\rangle \otimes |0\rangle - \frac{1}{2}|0\rangle \otimes |1\rangle - \frac{1}{2}|1\rangle \otimes |0\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle.$$

The first two terms have a common factor $|0\rangle$, the third and the fourth one have a common factor $|1\rangle$, so we have

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{\sqrt{2}}|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) - \frac{1}{\sqrt{2}}|1\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).$$

This expression can be equivalently reformulated as

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).$$

For the first bit, we have $H|1\rangle$, for the second bit, we also have $H|1\rangle$:

$$f(H(|0\rangle) \otimes H(|1\rangle)) = H(|1\rangle) \otimes H(|1\rangle).$$

It is known that when we apply the Hadamard transformation twice, we get the same stat. In particular, $H(H(|1\rangle)) = |1\rangle$. Thus, when we apply the Hadamard transformation to both bits once again, we get the state

$$|1\rangle \otimes |1\rangle.$$

Measuring the value of the first bit, we get the value 1 with probability $|1|^2 = 1$. Thus, we can indeed conclude that $f(0) \neq f(1)$ – and we called the function f only once.