

## Solution to Homework 37

**Problem.** What class of polynomial hierarchy contains  $\Sigma_2\text{P}^{\Pi_3\text{P}}$ ? Explain your answer.

**Solution.** For each oracle  $A$ , the class  $\Sigma_2\text{P}^A$  is described by formulas  $F$  with 2 quantifiers starting with the existential quantifier in which the main property is feasible with respect to  $A$ :

$$F \equiv \exists x_1 \forall x_2 C^A(x_1, x_2, x). \quad (1)$$

Here, the fact that  $C^{\Pi_3\text{P}}$  is feasible with respect to the corresponding oracle  $A = \Pi_3\text{P}$  means that this property, in its turn, has the form

$$\begin{aligned} C^A(x_1, x_2, x) &= C^{\Pi_3\text{P}}(x_1, x_2, x_3, x_4, x_5, x) \equiv \\ &\forall x_3 \exists x_4 \forall x_5 C(x_1, x_2, x_3, x_4, x_5, x), \end{aligned} \quad (2)$$

where the property  $C(x_1, x_2, x_3, x_4, x_5, x)$  is actually feasible.

Substituting the formula (2) into the expression (1), we get

$$F \equiv \exists x_1 \forall x_2 \forall x_3 \exists x_4 \forall x_5 C(x_1, x_2, x_3, x_4, x_5, x). \quad (3)$$

As mentioned in the lecture, two consequent quantifiers – such as  $\forall x_2 \forall x_3$  – is equivalent to  $\forall(x_2, x_3)$ , and can thus be compressed into a single quantifier  $\forall x'_2$  – since, as we have learned earlier, there is a feasible correspondence between natural numbers and pairs of natural numbers. In this correspondence, each code  $n$  of the pair can be transformed back into the pair  $(\pi_1(n), \pi_2(n))$ .

Thus, the formula (3) takes the simplified form

$$F \equiv \exists x_1 \forall x'_2 \exists x_4 \forall x_5 C(x_1, \pi_1(x'_2), \pi_2(x'_2), x_4, x_5, x).$$

In this expression, we have 4 quantifiers, the first of which is the existential quantifier. Thus, this formula belongs to the class  $\Sigma_4\text{P}$ , i.e.:

$$\Sigma_2\text{P}^{\Pi_3\text{P}} \subseteq \Sigma_4\text{P}.$$