Solution to Problem 3

**Problem.** Prove, from scratch – i.e., using only the definition of the primitive recursive function (and not using any results that we had in class without proving them) – that the function \( (a/b)/(a\%b) \) is primitive recursive.

**Solution.** To prove that the desired function is primitive recursive (p.r.), we will show:

- that remainder is p.r. and
- that division is p.r.

1. The remainder function \( \text{rem}(a, b) = b \% a \) can be represented as follows:
   \[
   \text{rem}(a, 0) = 0;
   \]
   \[
   \text{rem}(a, b + 1) = \text{if } (\text{rem}(a, b) + 1 < b) \text{ then } (\text{rem}(a, b) + 1) \text{ else } 0.
   \]

To show that this expression is p.r., we need to show:

- that \(<\) is p.r., and
- that the if-then-else construction is p.r.

1.1. Let us first show that the relation \( r < b \) is p.r. Indeed, the condition \( r < b \) is equivalent to \( b - r > 0 \), where:
   - \( b - r = b - r \) if \( b > r \) and
   - \( b - r = 0 \) otherwise.

So, to prove that \( r < b \) is p.r., it is sufficient to prove that the relation \( a > 0 \) and subtraction \( \text{sub}(b, r) = b - r \) are p.r.

1.1.1. To prove that \( a > 0 \) is p.r., we can use the following description of this relation:
   \[
   \text{pos}(0) = 0;
   \]
   \[
   \text{pos}(m + 1) = 1.
   \]

1.1.2. Subtraction can be represented as
   \[
   \text{sub}(b, 0) = b;
   \]
   1
\[ \text{sub}(b, r + 1) = \text{sub}(b, r) \div 1. \]

So, to show that subtraction is p.r., it is sufficient to prove the function \( \text{prev}(n) = n \div 1 \) is p.r.

1.1.3. Indeed, the function \( \text{prev}(n) \) can be represented as follows:

\[
\begin{align*}
\text{prev}(0) &= 0; \\
\text{prev}(n + 1) &= n.
\end{align*}
\]

1.2. Let us now show that the if-the-else construction is p.r. Indeed, the if-then-else construction can be represented as

\[
\text{if } (P(\pi)) \text{ then } f(\pi) \text{ else } g(\pi) = P(\pi) \cdot f(\pi) + (1 - P(\pi)) \cdot g(\pi).
\]

We already know that subtraction are p.r., so to prove that if-then-else construction is p.r., we need to prove that addition and multiplication are p.r.

1.2.1. Addition is p.r. since the function

\[
\text{add}(a, b) = a + b = a + 1 + \ldots + 1 \ (b \text{ times})
\]

can be represented as

\[
\begin{align*}
\text{add}(a, 0) &= a; \\
\text{add}(a, b + 1) &= \text{add}(a, b) + 1.
\end{align*}
\]

1.2.2. Multiplication is p.r. since it can be represented as

\[
\text{mult}(a, b) = a \cdot b = a + \ldots + a \ (b \text{ times}),
\]

thus

\[
\begin{align*}
\text{mult}(a, 0) &= 0; \\
\text{mult}(a, b + a) &= \text{mult}(a, b) + a.
\end{align*}
\]

1.3. So, if-then-else construction is p.r., thus remainder is p.r.

2. Let us now show that division is p.r. Indeed, division \( \text{div}(a, b) = b / a \) can be represented as follows:

\[
\begin{align*}
\text{div}(a, 0) &= 0; \\
\text{div}(a, b + 1) &= \text{if } (\text{rem}(a, b + 1) > 0) \text{ then } \text{div}(a, b) \text{ else } (\text{div}(a, b) + 1).
\end{align*}
\]

Here, if-then-else construction, remainder, and \( > \) are all p.r., so division is also p.r.

3. Since multiplication, remainder, and division are all p.r., the desired function – which is their composition – is also p.r.