Solution to Problem 4

Problem. Write a Java program corresponding to the following primitive recursive function $f = \sigma(\sigma(PR(\pi_2^2, \sigma(add(\pi_1^4, \pi_2^4, \pi_3^4)))))$. For this function f, what is the value of $f(1, 1, 1)$?

Solution. In general, the expression $h = PR(f, g)$ corresponding to functions $f(n_1, \ldots, n_k)$ and $g(n_1, \ldots, n_k, m, h)$ defines a function of $k + 1$ variables:

$$h(n_1, \ldots, n_k, 0) = f(n_1, \ldots, n_k);$$

$$h(n_1, \ldots, n_k, m + 1) = g(n_1, \ldots, n_k, m, h(n_1, \ldots, n_k, m)).$$

In our cases, $f = \pi_2^2$ is a function of 2 variables, so $k = 2$. For $k = 2$, the general formulas for primitive recursion take the following form:

$$h(n_1, n_2, 0) = f(n_1, n_2);$$

$$h(n_1, n_2, m + 1) = g(n_1, n_2, m, h(n_1, n_2, m)).$$

Here, $f(n_1, n_2) = \pi_2^2(n_1, n_2) = n_2$ and

$$g(n_1, n_2, m, h) = \sigma(add(\pi_1^4(n_1, n_2, m, h), \pi_2^4(n_1, n_2, m, h), \pi_3^4(n_1, n_2, m, h))) =$$

$$\sigma(add(n_1, n_2, m)) = n_1 + n_2 + m + 1.$$

Thus, we have

$$h(n_1, n_2, 0) = n_2;$$

$$h(n_1, n_2, m + 1) = n_1 + n_2 + m + 1.$$

Primitive recursion is the description of a for-loop. The first line of the primitive recursion describes what is happening before the loop. In Java, the corresponding statement takes the following form:

```
int h = n2;
```

The second line of the primitive recursion describes what happens when we get from the iteration number $i - 1 = m$ to iteration number $i = m + 1$. So, we take

```
h = n1 + n2 + i;
```

The whole code for the PR part takes the form:

```
int h = n2;
for(int i = 1; i <= m; i++)
  {h = n1 + n2 + i;}
```

The desired function f is obtained from the PR expression by applying σ twice. Thus, we have the following Java program for computing the function f:

```
int h = n2;
for(int i = 1; i <= m; i++)
  {h = n1 + n2 + i;}
h++;
h++;
```

Let us trace this Java program on the example of $n1 = 1$, $n2 = 1$, and $m = 1$.

- We start with assigning, to the variable h, the value $n1 = 1$.
- Then, we go into the for-loop, and define the new variable i whose value is 1.
- Here, $i = 1 \le m = 1$, so we go inside the loop, and assign, to the variable h, the new value $h = n1 + n2 + i = 1 + 1 + 1 = 3$.
- After that, we increase i by 1, so i is now 2.
- Here still, $i = 2 > m = 1$, so we get out of the loop.
- Finally, we twice increase the value h by 1, getting $h = 5$.

The value 5 is the desired value of the function $f(1, 1, 1)$.

Comment. Instead of tracing the Java program, we can trace the original formulas for primitive recursion, which for $n_1 = 1$ and $n_2 = 1$, take the form

$$h(1, 1, 0) = 1;$$

$$h(1, 1, m + 1) = 1 + 1 + m + 1.$$

For $m = 0$, the second formula leads to

$$h(1, 1, 1) = 1 + 1 + 0 + 1 = 3.$$

Thus, in this case, $h = PR(\ldots) = 4$.

To get the value of the desired function $f = \sigma(\sigma(PR(\ldots)))$, we need to add 2 to the PR expression $PR(\ldots) = 3$, so the final answer is 5.