Solution to Problem 8

Problem. Describe integer division a/b in terms of μ-recursion, similarly to how in the lecture, we describe subtraction in terms of μ-recursion.

Solution. What does it mean that $a/b = m$? It means that $a = b \cdot m + r$ for some remainder r – a non-negative number which is smaller than b: $0 \le r < b$.

By definition of the remainder, $r = a - b \cdot m$, so the above inequality means $0 \le a - b \cdot m < b$. To avoid using subtraction – which may result in a negative number, while we only consider non-negative integers – we can add $b \cdot m$ to all three sides of this inequality and get $b \cdot m \le a < b \cdot m + b$. So, we can define a/b as

$$a/b = \mu m.(b \cdot m \le a < b \cdot m + b).$$

Comment. This expression can be simplified. For example, in the expression $b \cdot m + b$, both terms have a common factor m, so this expression can be simplified into a simpler expression $b \cdot (m+1)$ which has adding 1 instead of a full addition. So, we get:

$$a/b = \mu m.(b \cdot m \le a < b \cdot (m + 1)).$$

In this form, one can see that actually m is the first value for which $a < b \cdot (m+1)$, since for the previous value $m' = m - 1$, we have $b \cdot (m' + 1) = b \cdot m \le a$. Thus, we can write that

$$a/b = \mu m.(a < b \cdot (m + 1)).$$