Solution to Problem 15

**Problem.** Sketch an example of a Turing machine for implementing μ-recursion, the way we did it in class, on the example of a function \( \mu m. (m \geq a) \).

**Solution.** The given function is a particular case of a general \( \mu \)-recursion expression

\[
f(n_1, \ldots, n_k) = \mu m.P(n_1, \ldots, n_k, m)
\]
corresponding to \( k = 1 \) and \( P(n_1, m) \iff n_1 \geq m \).

Suppose that we have a Turing machine for computing the inequality relation \( P(n_1, m) \). According to the general algorithm described in the lecture, we start with the state

\[
\_ n_1 \_ \_ \ldots \text{start}
\]
and we want to end up in the state

\[
\_ f(n_1) \_ \_\ldots \text{halt}
\]

Let us show how this can be done. First, we add 0 after the input, copy the whole tuple \((n_1, 0)\), and move the head before the second copy of \( n_1 \):

\[
\_ n_1 \_ 0 \_ n_1 \_ 0 \_ \ldots
\]

Then, we apply the Turing machine computing the function \( P(n_1, 0) \). As a result, we get the following state:

\[
\_ n_1 \_ 0 \_ P(n_1, 0) \_ \ldots
\]

If \( P(n_1, 0) = 0 \) (i.e., if the property \( P(n_1, m) \) is false), then we increase 0 by 1, copy the tuple \((n_1, 1)\):

\[
\_ n_1 \_ 1 \_ n_1 \_ 1 \_ \ldots
\]

and again apply the Turing machine for computing \( P(n_1, m) \), resulting in:

\[
\_ n_1 \_ 1 \_ P(n_1, 1) \_ \ldots
\]

In general, at each iteration, we start with the state

\[
\_ n_1 \_ m \_ P(n_1, m) \_ \ldots
\]

If \( P(n_1, m) = 0 \) (i.e., to “false”), then we increase \( m \) by 1, copy the tuple \((n_1, m + 1)\):

\[
\_ n_1 \_ m + 1 \_ n_1 \_ m + 1 \_ \ldots
\]
and again apply the Turing machine for computing $P(n_1, m + 1)$, resulting in:

$$\begin{array}{cccccc}
- & n_1 & - & m + 1 & \leq & P(n_1, m + 1) & - & \ldots
\end{array}$$

e tc.

This continues until we get the first value $m$ for which $P(n_1, m) = 1$ (i.e., “true”). In this case, we get the state

$$\begin{array}{cccccc}
- & n_1 & - & m & \leq & 1 & - & \ldots
\end{array}$$

Here, the desired value $m$ is 2-nd out of 3, so it can be found if we apply the corresponding projection $\pi_3^2$, resulting in:

$$\begin{array}{cccc}
- & m & - & \ldots
\end{array} \text{halt}$$

where $m = f(n_1) = \mu m. P(n_1, m)$.

This is exactly what we wanted.