Solution to Problem 35

**Problem.** Use the variable-elimination algorithm for checking satisfiability of 2-SAT formulas that we had in class to find the values that satisfy the following formula:

\[(\neg a \lor \neg b) \land (a \lor \neg \neg b) \land (a \lor \neg c) \land (\neg a \lor \neg c) \land (b \lor \neg c).\]

**Solution.** In this formula, we have three Boolean variables: \(a\), \(b\), and \(c\). According to the algorithm, we need to eliminate them one by one.

Let us first the variable \(a\). According to the general algorithm:

- each clause of the type \(a \lor x\) is converted to an inequality \(\neg x \leq a\), and
- each clause of the type \(\neg a \lor x\) is converted into an inequality \(a \leq x\).

Thus, in the above formula, clauses containing \(a\) or \(\neg a\) are converted into the following inequalities:

- the clause \(\neg a \lor \neg b\) is converted into an inequality \(a \leq \neg b\);
- the clause \(a \lor b\) is converted into an inequality \(\neg b \leq a\);
- the clause \(a \lor \neg b\) is converted into an inequality \(b \leq a\);
- the clause \(a \lor \neg c\) is converted into an inequality \(c \leq a\); and
- the clause \(\neg a \lor \neg c\) is converted into an inequality \(a \leq \neg c\).

Here, we have:

- three lower bounds for \(a\): \(\neg b \leq a\), \(b \leq a\), and \(c \leq a\), and
- two upper bounds for \(a\): \(a \leq \neg b\) and \(a \leq \neg c\).

In other words, we have:

\[\neg b, b, c \leq a \leq \neg b, \neg c.\]  

(1)

Each lower bound must be smaller than or equal to each upper bound. So, we get the following inequalities:

- from the first lower bound, we get \(\neg b \leq \neg b\) and \(\neg b \leq \neg c\);
• from the second lower bound, we get $b \leq \neg b$ and $b \leq \neg c$;

• from the third lower bound, we get $c \leq \neg b$ and $c \leq \neg c$;

As in the example given in the lecture, the inequality $b \leq \neg b$ is only true when $b = 0$. Similarly, the inequality $c \leq \neg c$ is only satisfied when $c = 0$. For these values $b$ and $c$, the inequality (1) takes the form

$$1, 0, 0 \leq a \leq 1, 1$$

i.e., the form $1 \leq a \leq 1$. Thus, $a = 1$.

So, the solution is: $a = 1$, $b = 0$, and $c = 0$. In other words, $a$ is true, and $b$ and $c$ is false.