Solution to Problem 15

Problem. Sketch an example of a Turing machine for implementing mu-recursion, the way we did it in class, on the example of a function $\mu m.(m = a)$.

Solution. The given function is a particular case of a general $\mu$-recursion expression

$$f(n_1, \ldots, n_k) = \mu m.P(n_1, \ldots, n_k, m)$$

corresponding to $k = 1$ and $P(n_1, m) \leftrightarrow n_1 = m$.

Suppose that we have a Turing machine for computing the inequality relation $P(n_1, m)$. According to the general algorithm described in the lecture, we start with the state

$$\_n_1\_ \ldots \text{start}$$

and we want to end up in the state

$$\_ f(n_1) \_ \ldots \text{halt}$$

Let us show how this can be done. First, we add 0 after the input, copy the whole tuple $(n_1, 0)$, and move the head before the second copy of $n_1$:

$$\_n_1\_0\_n_1\_0\_ \ldots$$

Then, we apply the Turing machine computing the function $P(n_1, 0)$. As a result, we get the following state:

$$\_n_1\_0\_P(n_1, 0)\_ \ldots$$

If $P(n_1, 0) = 0$ (i.e., the property $P(n_1, m)$ is false), then we increase 0 by 1, copy the tuple $(n_1, 1)$:

$$\_n_1\_1\_n_1\_1\_ \ldots$$

and again apply the Turing machine for computing $P(n_1, m)$, resulting in:

$$\_n_1\_1\_P(n_1, 1)\_ \ldots$$

In general, at each iteration, we start with the state

$$\_n_1\_m\_P(n_1, m)\_ \ldots$$

If $P(n_1, m) = 0$ (i.e., to “false”), then we increase $m$ by 1, copy the tuple $(n_1, m + 1)$:

$$\_n_1\_m+1\_n_1\_m+1\_ \ldots$$
and again apply the Turing machine for computing $P(n_1, m + 1)$, resulting in:

\[ n_1 \quad m + 1 \quad P(n_1, m + 1) \quad \ldots \]

etc.

This continues until we get the first value $m$ for which $P(n_1, m) = 1$ (i.e., “true”). In this case, we get the state:

\[ n_1 \quad m \quad 1 \quad \ldots \]

Here, the desired value $m$ is 2-nd out of 3, so it can be found if we apply the corresponding projection $\pi^3_2$, resulting in:

\[ m \quad \ldots \quad \text{halt} \]

where $m = f(n_1) = \mu m.P(n_1, m)$.

This is exactly what we wanted.