Solution to Problem 1

Problem 1. Prove that the function computing the product

$$(1^2 + 1) \cdot (2^2 + 1) \cdot (3^2 + 1) \cdot \ldots \cdot (n^2 + 1)$$

is primitive recursive. This proof should follow the same pattern that we used in class to prove that addition and multiplication are primitive recursive:

- You start with a 3-dot expression.
- First you write a for-loop corresponding to this function
- Then you describe this for-loop in mathematical terms
- Then, to prepare for a match with the general expression for primitive recursion, you rename the function to f and the parameters to $n_1, \ldots, n_k, m$
- Then you write down the general expression of primitive recursion for the corresponding k
- Then you match: find f and g for which the specific case of primitive recursion will be exactly the functions corresponding to initialization and to what is happening inside the loop
- Then, you get a final expression for the function

  $$(1^2 + 1) \cdot (2^2 + 1) \cdot (3^2 + 1) \cdot \ldots \cdot (n^2 + 1)$$

  that proves that this function is primitive recursive, i.e., that it can be formed from 0, π^k_i, and σ by composition and primitive recursion.

Solution. Here is the for-loop for computing the desired expression:

```
int v = 1;
for (int i = 1; i <= n; i++){
  v = v * (i * i + 1);}
```

Let us now describe this for-loop in mathematical terms. At each iteration i, we multiply the value by $i^2 + 1$. The value $v(i)$ is the value of the variable v

after iteration i. So, $v(m+1)$ is the value of the variable v after iteration $m+1$. In this case, $i = m+1$, so we get:

$$v(0) = 1;$$

$$v(m+1) = v(m) \cdot ((m+1) \cdot (m+1) + 1).$$

To prepare for the match, we rename the function to h (here, there are no other parameters to rename):

$$h(0) = 0;$$

$$h(m+1) = h(m) \cdot ((m+1) \cdot (m+1) + 1).$$

Here, we are defining a function of 1 variable. In general, primitive recursion defines a function of $k+1$ variables. Here, $k+1 = 1$, so $k = 0$, and the general expression for primitive recursion takes the following form:

$$h(0) = f;$$

$$h(m+1) = g(m, h(m)).$$

To match with the above description, we need to take $f = 1 = \sigma \circ 0$ and $g(m,h) = h * ((m+1) * (m+1) + 1)$, i.e., $g = prod(\pi^2_2, \sigma(prod(\sigma \circ \pi^2_1, \sigma \circ \pi^2_1)))$.

Thus, the desired expression for our function is

$$PR(\sigma \circ 0, prod(\pi^2_2, \sigma(prod(\sigma \circ \pi^2_1, \sigma \circ \pi^2_1)))).$$