Solution to Problem 1

**Problem 1.** Prove that the function computing the product

\[(1^2 + 1) \cdot (2^2 + 1) \cdot (3^2 + 1) \cdot \ldots \cdot (n^2 + 1)\]

is primitive recursive. This proof should follow the same pattern that we used in class to prove that addition and multiplication are primitive recursive:

- You start with a 3-dot expression.
- First you write a for-loop corresponding to this function
- Then you describe this for-loop in mathematical terms
- Then, to prepare for a match with the general expression for primitive recursion, you rename the function to \(f\) and the parameters to \(n_1, \ldots, n_k, m\)
- Then you write down the general expression of primitive recursion for the corresponding \(k\)
- Then you match: find \(f\) and \(g\) for which the specific case of primitive recursion will be exactly the functions corresponding to initialization and to what is happening inside the loop
- Then, you get a final expression for the function

\[(1^2 + 1) \cdot (2^2 + 1) \cdot (3^2 + 1) \cdot \ldots \cdot (n^2 + 1)\]

that proves that this function is primitive recursive, i.e., that it can be formed from 0, \(\pi^k\), and \(\sigma\) by composition and primitive recursion.

**Solution.** Here is the for-loop for computing the desired expression:

```java
int v = 1;
for (int i = 1; i <= n; i++){
    v = v * (i * i + 1);
}
```

Let us now describe this for-loop in mathematical terms. At each iteration \(i\), we multiply the value by \(i^2 + 1\). The value \(v(i)\) is the value of the variable \(v\)
after iteration $i$. So, $v(m + 1)$ is the value of the variable $v$ after iteration $m + 1$. In this case, $i = m + 1$, so we get:

$$v(0) = 1;$$

$$v(m + 1) = v(m) \cdot ((m + 1) \cdot (m + 1) + 1).$$

To prepare for the match, we rename the function to $h$ (here, there are no other parameters to rename):

$$h(0) = 0;$$

$$h(m + 1) = h(m) \cdot ((m + 1) \cdot (m + 1) + 1).$$

Here, we are defining a function of 1 variable. In general, primitive recursion defines a function of $k + 1$ variables. Here, $k + 1 = 1$, so $k = 0$, and the general expression for primitive recursion takes the following form:

$$h(0) = f;$$

$$h(m + 1) = g(m, h(m)).$$

To match with the above description, we need to take $f = 1 = \sigma \circ 0$ and $g(m, h) = h \star ((m + 1) \star (m + 1) + 1)$, i.e., $g = \text{prod}(\pi_2^2, \sigma(\text{prod}(\sigma \circ \pi_2^1, \sigma \circ \pi_2^1))).$

Thus, the desired expression for our function is

$$PR(\sigma \circ 0, \text{prod}(\pi_2^2, \sigma(\text{prod}(\sigma \circ \pi_2^1, \sigma \circ \pi_2^1)))).$$