Solution to Problem 23

**Problem.** Logarithms were invented to make multiplication and division faster: they reduce:
- multiplication to addition and
- division to subtraction

by using the formulas
\[
\ln(x_1 \cdot x_2) = \ln(x_1) + \ln(x_2) \quad \text{and} \quad \ln\left(\frac{x_1}{x_2}\right) = \ln(x_1) - \ln(x_2).
\]
Specifically:
- instead of directly computing the product \( y = x_1 \cdot x_2 \), we first compute \( X_1 = \ln(x_1) \) and \( X_2 = \ln(x_2) \), then compute \( Y = X_1 + X_2 \), and then \( y = \exp(Y) \);
- instead of directly computing the ratio \( y = \frac{x_1}{x_2} \), we first compute \( X_1 = \ln(x_1) \) and \( X_2 = \ln(x_2) \), then compute \( Y = X_1 - X_2 \), and then \( y = \exp(Y) \).

Describe each of these two reductions in general terms: what is \( C(x, y) \), what is \( C'(x', y') \), what is \( U_1 \), \( U_2 \), and \( U_3 \).

**Solution for the product.** Here \( x = (x_1, x_2) \), \( C(x, y) \) is the desired property \( y = x_1 \cdot x_2 \). For the problem to which we reduce, we have \( x' = (X_1, X_2) \), \( y' = Y \), and the property \( C'(x', y') \) is \( Y = X_1 + X_2 \).

Here, \( U_1(x_1, x_2) = (\ln(x_1), \ln(x_2)) \), \( U_2(Y) = \exp(Y) \), and \( U_3(y) = \ln(y) \).

**Solution for the ratio.** Here \( x = (x_1, x_2) \), \( C(x, y) \) is the desired property \( y = \frac{x_1}{x_2} \). For the problem to which we reduce, we have \( x' = (X_1, X_2) \), \( y' = Y \), and the property \( C'(x', y') \) is \( Y = X_1 - X_2 \).

Here, \( U_1(x_1, x_2) = (\ln(x_1), \ln(x_2)) \), \( U_2(Y) = \exp(Y) \), and \( U_3(y) = \ln(y) \).