Solution to Problem 23

Problem. Logarithms were invented to make multiplication and division faster: they reduce:

- multiplication to addition and
- division to subtraction

by using the formulas

$$\ln(x_1 \cdot x_2) = \ln(x_1) + \ln(x_2) \text{ and } \ln(x_1/x_2) = \ln(x_1) - \ln(x_2).$$

Specifically:

- instead of directly computing the product $y = x_1 \cdot x_2$, we first compute $X_1 = \ln(x_1)$ and $X_2 = \ln(x_2)$, then compute $Y = X_1 + X_2$, and then $y = \exp(Y)$;
- instead of directly computing the ratio $y = x_1/x_2$, we first compute $X_1 = \ln(x_1)$ and $X_2 = \ln(x_2)$, then compute $Y = X_1 - X_2$, and then $y = \exp(Y)$.

Describe each of these two reductions in general terms: what is $C(x, y)$, what is $C'(x', y')$, what is U_1, U_2, and U_3.

Solution for the product. Here $x = (x_1, x_2)$, $C(x, y)$ is the desired property $y = x_1 \cdot x_2$. For the problem to which we reduce, we have $x' = (X_1, X_2)$, $y' = Y$, and the property $C'(x', y')$ is $Y = X_1 + X_2$.

Here, $U_1(x_1, x_2) = (\ln(x_1), \ln(x_2))$, $U_2(Y) = \exp(Y)$, and $U_3(y) = \ln(y)$.

Solution for the ratio. Here $x = (x_1, x_2)$, $C(x, y)$ is the desired property $y = x_1/x_2$. For the problem to which we reduce, we have $x' = (X_1, X_2)$, $y' = Y$, and the property $C'(x', y')$ is $Y = X_1 - X_2$.

Here, $U_1(x_1, x_2) = (\ln(x_1), \ln(x_2))$, $U_2(Y) = \exp(Y)$, and $U_3(y) = \ln(y)$.