Solution to Problem 2

Problem 2. Prove that if $P(\overline{n})$, $Q(\overline{n})$, $R(\overline{n})$, $f(\overline{n})$, $g(\overline{n})$, $h(\overline{n})$, and $i(\overline{n})$ are all primitive recursive, then the function described as

$$F(\overline{n}) = \text{if (not } P(\overline{n})\text{) then } f(\overline{n}) \text{ elseif (not } Q(\overline{n})\text{) then } g(\overline{n}) \text{ elseif (not } R(\overline{n})\text{)}$$
$$\text{then } h(\overline{n}) \text{ else } i(\overline{n})$$

is also primitive recursive.

Solution. **Proof.** We want:

- the value $f(\overline{n})$ if $P(\overline{n})$ is false, i.e., if the expression

  $$\text{not } P(\overline{n})$$

  is true, i.e., equivalently, if $1 \dot{-} P(\overline{n})) = 1$;

- the value $g(\overline{n})$ if $P(\overline{n})$ is true and $Q(\overline{n})$ is false, i.e., if the expression

  $$P(\overline{n}) \text{ and (not } Q(\overline{n})\text{)}$$

  is true, i.e., equivalently, if $P(\overline{n}) \cdot (1 \dot{-} Q(\overline{n})) = 1$;

- the value $h(\overline{n})$ if $P(\overline{n})$ and $Q(\overline{n})$ are true and $R(\overline{n})$ is false, i.e., if the expression

  $$P(\overline{n}) \text{ and } Q(\overline{n}) \text{ and (not } R(\overline{n})\text{)}$$

  is true, i.e., equivalently, if $P(\overline{n}) \cdot Q(\overline{n}) \cdot (1 \dot{-} R(\overline{n})) = 1$; and

- the value $h(\overline{n})$ if all three statements $P(\overline{n})$, $Q(\overline{n})$ and $R(\overline{n})$ are true, i.e., if the expression

  $$P(\overline{n}) \text{ and } Q(\overline{n}) \text{ and } R(\overline{n})$$

  is true, i.e., equivalently, if $P(\overline{n}) \cdot Q(\overline{n}) \cdot R(\overline{n}) = 1$.

Thus, it makes sense to take

$$F(\overline{n}) = (1 \dot{-} P(\overline{n})) \cdot f(\overline{n}) + P(\overline{n}) \cdot (1 \dot{-} Q(\overline{n})) \cdot g(\overline{n}) + P(\overline{n}) \cdot Q(\overline{n}) \cdot (1 \dot{-} R(\overline{n})) \cdot h(\overline{n}) +$$
$$P(\overline{n}) \cdot Q(\overline{n}) \cdot R(\overline{n}) \cdot i(\overline{n}).$$

Then:

- if $P(\overline{n})$ is false, i.e., if $P(\overline{n}) = 0$, then $1 \dot{-} P(\overline{n}) = 1$, so

$$F(\overline{n}) = 1 \cdot f(\overline{n}) + 0 \cdot g(\overline{n}) + 0 \cdot h(\overline{n}) + 0 \cdot i(\overline{n}) = f(\overline{n});$$

- if $P(\overline{n})$ is true and $Q(\overline{n})$ is false, then $1 \dot{-} P(\overline{n}) = 0$ and $1 \dot{-} Q(\overline{n}) = 1$, then

$$F(\overline{n}) = 0 \cdot f(\overline{n}) + 1 \cdot g(\overline{n}) + 0 \cdot h(\overline{n}) + 0 \cdot i(\overline{n}) = g(\overline{n});$$

- if $P(\overline{n})$ and $Q(\overline{n})$ are both true and $R(\overline{n})$ is false, then $1 \dot{-} P(\overline{n}) = 1 \dot{-} Q(\overline{n}) = 0$ and $1 \dot{-} R(\overline{n}) = 1$, then

$$F(\overline{n}) = 0 \cdot f(\overline{n}) + 0 \cdot g(\overline{n}) + 1 \cdot h(\overline{n}) + 0 \cdot i(\overline{n}) = h(\overline{n});$$

- if all three statements $P(\overline{n})$, $Q(\overline{n})$, and $R(\overline{n})$ are true, then $1 \dot{-} P(\overline{n}) = 1 \dot{-} Q(\overline{n}) = 1 \dot{-} R(\overline{n}) = 0$, then

$$F(\overline{n}) = 0 \cdot f(\overline{n}) + 0 \cdot g(\overline{n}) + 0 \cdot h(\overline{n}) + 1 \cdot i(\overline{n}) = i(\overline{n}).$$