

Solution to Problem 36

Problem. On the example of the function $f(x) = x$, trace, step by step, how Deutsch-Jozsa algorithm will conclude that $f(0) \neq f(1)$ while applying f only once.

Solution. The Deutsch-Jozsa algorithm works as follows:

- we start with the state $|0, 1\rangle = |0\rangle \otimes |1\rangle$;
- we apply the Hadamard transformation H to both bits;
- then, we apply f ;
- after that, we again apply the Hadamard transformation to both bits;
- finally, we measure the first bit of the resulting 2-bit state:
 - if the first bit is 0, we conclude that the function f is constant;
 - if the first bit is 1, we conclude that the function f is not constant.

We start with the state $|0, 1\rangle = |0\rangle \otimes |1\rangle$, in which the first bit is in state $|0\rangle$ and the second bit is in state $|1\rangle$. When we apply the Hadamard transformation to both bits, we get

$$H(|0\rangle) \otimes H(|1\rangle) = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right).$$

Due to linearity, we get

$$H(|0\rangle) \otimes H(|1\rangle) = \frac{1}{2}|0, 0\rangle - \frac{1}{2}|0, 1\rangle + \frac{1}{2}|1, 0\rangle - \frac{1}{2}|1, 1\rangle.$$

Then, we apply the formula $|x, y\rangle \mapsto |y \oplus f(x)\rangle$. In our case, $f(0) = 0$ and $f(1) = 1$, so

$$f(|0, 0\rangle) = |0, 0\rangle, \quad f(|0, 1\rangle) = |0, 1\rangle, \quad f(|1, 0\rangle) = |1, 1\rangle, \quad f(|1, 1\rangle) = |1, 0\rangle,$$

and, thus,

$$\begin{aligned} f(H(|0\rangle) \otimes H(|1\rangle)) &= \frac{1}{2}|0, 0\rangle - \frac{1}{2}|0, 1\rangle + \frac{1}{2}|1, 1\rangle - \frac{1}{2}|1, 0\rangle = \\ &= \frac{1}{2}|0\rangle \otimes |0\rangle - \frac{1}{2}|0\rangle \otimes |1\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle - \frac{1}{2}|1\rangle \otimes |0\rangle. \end{aligned}$$

The first two terms have a common factor $|0\rangle$, the third and the fourth one have a common factor $|1\rangle$, so we have

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{\sqrt{2}}|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right) + \frac{1}{\sqrt{2}}|1\rangle \otimes \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle \right).$$

This expression can be equivalently reformulated as

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{\sqrt{2}}|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) - \frac{1}{\sqrt{2}}|1\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right),$$

and thus, as

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right).$$

For both bits, what we have is $H|1\rangle$. Thus, when we apply the Hadamard transformation once again, we get $|1, 1\rangle$.

So, when we measure the first bit, we get 1 with the probability $|1|^2 = 1$. Thus, the algorithm confirms that $f(0) \neq f(1)$.