Solution to Homework 37

Problem. What class of polynomial hierarchy contains $\Sigma_2\mathrm{P}^{\Sigma_2\mathrm{P}}$? Explain your answer.

Solution. For each oracle A, the class $\Sigma_2\mathrm{P}^A$ is described by formulas F with 2 quantifiers starting with the existential quantifier in which the main property is feasible with respect to A:

$$F \equiv \exists x_1 \, \forall x_2 \, C^A(x_1, x_2, x). \tag{1}$$

Here, the fact that $C^{\Sigma_2\mathrm{P}}$ is feasible with respect to the corresponding oracle $A = \Sigma_2\mathrm{P}$ means that this property, in its turn, has the form

$$C^A(x_1, x_2, x) = C^{\Sigma_2\mathrm{P}}(x_1, x_2, x) \equiv$$

$$\exists x_3 \, \forall x_4 \, C(x_1, x_2, x_3, x_4, x), \tag{2}$$

where the property $C(x_1, x_2, x_3, x_4, x)$ is actually feasible.

Substituting the formula (2) into the expression (1), we get

$$F \equiv \exists x_1 \, \forall x_2 \, \exists x_3 \, \forall x_4 \, C(x_1, x_2, x_3, x_4, x). \tag{3}$$

In this expression, we have 4 quantifiers, the first of which is the existential quantifier. Thus, this formula belongs to the class $\Sigma_4\mathrm{P}$, i.e.:

$$\Sigma_2\mathrm{P}^{\Sigma_2\mathrm{P}} \subseteq \Sigma_4\mathrm{P}.$$