Problem. What class of polynomial hierarchy contains $\Sigma_2^P$? Explain your answer.

Solution. For each oracle $A$, the class $\Sigma_2^P$ is described by formulas $F$ with 2 quantifiers starting with the existential quantifier in which the main property is feasible with respect to $A$:

$$F \equiv \exists x_1 \forall x_2 C^A(x_1, x_2, x). \quad (1)$$

Here, the fact that $C^{\Sigma_2^P}$ is feasible with respect to the corresponding oracle $A = \Sigma_2^P$ means that this property, in its turn, has the form

$$C^A(x_1, x_2, x) = C^{\Sigma_2^P}(x_1, x_2, x) \equiv$$

$$\exists x_3 \forall x_4 C(x_1, x_2, x_3, x_4, x), \quad (2)$$

where the property $C(x_1, x_2, x_3, x_4, x)$ is actually feasible.

Substituting the formula (2) into the expression (1), we get

$$F \equiv \exists x_1 \forall x_2 \exists x_3 \forall x_4 C(x_1, x_2, x_3, x_4, x). \quad (3)$$

In this expression, we have 4 quantifiers, the first of which is the existential quantifier. Thus, this formula belongs to the class $\Sigma_4^P$, i.e.:

$$\Sigma_4^P \subseteq \Sigma_4^P.$$