Solution to Problem 1

Problem 1. Prove that the function computing the sum

$$(1 \cdot 1 + 1) + (2 \cdot 2 + 2) + (3 \cdot 3 + 3) + \ldots + (n \cdot n + n)$$

is primitive recursive. This proof should follow the same pattern that we used in class to prove that addition and multiplication are primitive recursive:

- You start with a 3-dot expression.
- First you write a for-loop corresponding to this function
- Then you describe this for-loop in mathematical terms
- Then, to prepare for a match with the general expression for primitive recursion, you rename the function to f and the parameters to n_1, $\ldots$, n_k, m
- Then you write down the general expression of primitive recursion for the corresponding k
- Then you match: find f and g for which the specific case of primitive recursion will be exactly the functions corresponding to initialization and to what is happening inside the loop
- Then, you get a final expression for the function

  $$(1 \cdot 1 + 1) + (2 \cdot 2 + 2) + (3 \cdot 3 + 3) + \ldots + (n \cdot n + n)$$

  that proves that this function is primitive recursive, i.e., that it can be formed from 0, π_i^k, and σ by composition and primitive recursion.

Solution. Here is the for-loop for computing the desired expression:

```
int v = 0;
for (int i = 1; i <= n; i++){
  v = v + (i * i + i);}
```

Let us now describe this for-loop in mathematical terms. At each iteration i, we multiply the value by $i \cdot i + i$. The value $v(i)$ is the value of the variable v

after iteration i. So, $v(m+1)$ is the value of the variable v after iteration $m+1$. In this case, $i = m+1$, so we get:

$$v(0) = 0;$$

$$v(m+1) = v(m) + ((m+1) \cdot (m+1) + (m+1)).$$

To prepare for the match, we rename the function to h (here, there are no other parameters to rename):

$$h(0) = 0;$$

$$h(m+1) = h(m) + ((m+1) \cdot (m+1) + (m+1)).$$

Here, we are defining a function of 1 variable. In general, primitive recursion defines a function of $k+1$ variables. Here, $k+1 = 1$, so $k = 0$, and the general expression for primitive recursion takes the following form:

$$h(0) = f;$$

$$h(m+1) = g(m, h(m)).$$

To match with the above description, we need to take $f = 0$ and $g(m, h) = h + ((m+1) * (m+1) + (m+1))$, i.e.,

$$g = sum(\pi^2_2, sum(prod(\sigma \circ \pi^2_1, \sigma \circ \pi^2_1), \sigma \circ \pi^2_1)).$$

Thus, the desired expression for our function is

$$PR(0, sum(\pi^2_2, sum(prod(\sigma \circ \pi^2_1, \sigma \circ \pi^2_1), \sigma \circ \pi^2_1))).$$