Solution to Problem 12

Problem. In class, we described a Turing machine that computes $g(n) = n + 1$ for unary n. In Homework 9, you designed a Turing machine that computes a function $f(n)$ which is equal to $n - 2$ when $n > 2$ and to 0 otherwise – for the case when n is unary.

In class, we described the general algorithm for designing a Turing machine that computes the composition of two functions. The assignment is to use this general algorithm to design a Turing machine that computes the composition $g(f(n))$. Trace, step-by-step, on an example, how your Turing machine works. For example, you can take as input $n = 1$.

Reminder: The Turing machine for computing $g(n) = n + 1$ for a unary input n is based on the following idea:

- we go step-by-step until we find the first blank space,
- then, we replace this blank space with 1 and go back.

This machine has the following rules:

- start, – → R, working
  (we start going to the right)
- working, 1 → R
  (we see 1, so we continue going),
- working, – → 1, L, back
  (we see a blank space, so we replace it with 1 and start going back)
- back, → L
  (while we see 1s, we continue going back)
- back, – → halt
  (once we reach the very first cell, we stop).

The Turing machine for computing $f(n)$ for unary n is based on the following idea:

- we go step-by-step until we find the first blank space,
- then, we go back, replace the last two 1s with blanks, and go back all the way.

We need to take special care of the case when $n < 2$.

Solution. The resulting Turing machine takes the following form:

- start, – $\to$ R, moving$_1$
- moving$_1$, – $\to$ L, start$_2$
- moving$_1$, 1 $\to$ R
- moving$_1$, – $\to$ 1, L, back$_1$
- back$_1$, 1 $\to$ L
- back$_1$, – $\to$ start$_2$
- start$_2$, – $\to$ R, working$_2$
- working$_2$, 1 $\to$ R
- working$_2$, – $\to$ L, delete1st$_2$
- delete1st$_2$, 1 $\to$ –, L, delete2nd$_2$
- delete1st$_2$, – $\to$ halt
- delete2nd$_2$, 1 $\to$ –, L, back
- deket2nd$_2$, – $\to$ halt
- back$_2$, 1 $\to$ L
- back$_2$, – $\to$ halt

Let us trace it for $n = 1$;

$\underline{-}$	1	–	–	–	–	–	...	start
–	$\underline{1}$	–	–	–	–	–	...	moving$_1$
–	1	$\underline{-}$	–	–	–	–	...	moving$_1$
–	$\underline{1}$	1	–	–	–	–	...	back$_1$
$\underline{-}$	1	1	–	–	–	–	...	back$_1$
$\underline{-}$	1	1	–	–	–	–	...	start$_2$
–	$\underline{1}$	1	–	–	–	–	...	working$_2$
–	1	$\underline{1}$	–	–	–	–	...	working$_2$
–	1	1	$\underline{-}$	–	–	–	...	working$_2$
–	1	$\underline{1}$	–	–	–	–	...	delete1st$_2$
–	$\underline{1}$	–	–	–	–	–	...	delete2nd$_2$
$\underline{-}$	–	–	–	–	–	–	...	halt