Solution to Problem 21

Problem. Let us consider cases when the sets A and B are decidable and the complement to the set C is r.e. Give four examples of such cases:

- an example when the union $A \cup B \cup C$ of the three sets A, B, and C is decidable,
- an example when the union $A \cup B \cup C$ of the three sets A, B, and C is not decidable,
- an example when the intersection of the three sets A, B, and C is decidable, and
- an example when the intersection of the three sets A, B, and C is not decidable.

Solution. Finding decidable examples is easy: it is sufficient to have decidable sets A, B, and C. For example:

- if $A = B = C = \emptyset$, then their union $A \cup B \cup C$ is the empty set and thus, decidable;
- if $A = B = C = \emptyset$, then their intersection $A \cap B \cap C$ is the empty set and thus, decidable.

Non-decidable examples are not so easy. To find such examples, let us recall that the union and intersection of decidable sets is decidable. Since the sets A and B are decidable, we need to find the set C which is not decidable. The only set for which we know that its complement is not decidable (it is even not r.e.) is the halting set. So, as C let us take the complement $-H$ to the halting set. Here is one of the possible ways to find such examples:

- if $A = B = \emptyset$ and $C = -H$, then their union $A \cup B \cup C$ is equal to $-H$ and is, thus, not decidable;
- if $A = B = N$ (the set of all natural numbers) and $C = -H$, then their intersection $A \cap B \cap C$ is equal to $-H$ and is, thus, not decidable.