Solution to Problem 29

Problem. Show how to compute the "and" of 15 boolean values in parallel if we have unlimited number of processors. How many processors do we need and how much time will the computation take? Why do we need parallel processing in the first place?

Solution. Suppose that we are given 15 boolean values $x_1, \ldots, x_{15}$, and we want to compute their conjunction ("and"). Then:

- at the first moment of time:
  - the first computer computes $x_1 \,\&\, x_2$,
  - the second computer computes $x_3 \,\&\, x_4$,
  - the third computer computes $x_5 \,\&\, x_6$,
  - the forth computer computes $x_7 \,\&\, x_8$,
  - the fifth computer computes $x_9 \,\&\, x_{10}$;
  - the sixth computer computes $x_{11} \,\&\, x_{12}$;
  - the seventh computer computes $x_{13} \,\&\, x_{14}$.
- at the second moment of time:
  - the first computer computes $x_1 \,\&\, x_2 \,\&\, x_3 \,\&\, x_4$ as
    $$(x_1 \,\&\, x_2) \,\&\, (x_3 \,\&\, x_4);$$
  - the second computer computes $x_5 \,\&\, x_6 \,\&\, x_7 \,\&\, x_8$ as
    $$(x_5 \,\&\, x_6) \,\&\, (x_7 \,\&\, x_8);$$
  - the third computer computes $x_9 \,\&\, x_{10} \,\&\, x_{11} \,\&\, x_{12}$ as
    $$(x_9 \,\&\, x_{10}) \,\&\, (x_{11} \,\&\, x_{12});$$
  - the fourth computer computes $x_{13} \,\&\, x_{14} \,\&\, x_{15}$ as
    $$(x_{13} \,\&\, x_{14}) \,\&\, x_{15}.$$
- at the third moment of time:

– the first computer computes

$$x_1 \,\&\, x_2 \,\&\, x_3 \,\&\, x_4 \,\&\, x_5 \,\&\, x_6 \,\&\, x_7 \,\&\, x_8$$

as

$$(x_1 \,\&\, x_2 \,\&\, x_3 \,\&\, x_4) \,\&\, (x_5 \,\&\, x_6 \,\&\, x_7 \,\&\, x_8);$$

– the first computer computes

$$x_9 \,\&\, x_{10} \,\&\, x_{11} \,\&\, x_{12} \,\&\, x_{13} \,\&\, x_{14} \,\&\, x_{15}$$

as

$$(x_9 \,\&\, x_{10} \,\&\, x_{11} \,\&\, x_{12}) \,\&\, (x_{13} \,\&\, x_{14} \,\&\, x_{15}).$$

- at the fourth moment of time, the first computer computes the desired value

  $x_1 \,\&\, x_2 \,\&\, x_3 \,\&\, x_4 \,\&\, x_5 \,\&\, x_6 \,\&\, x_7 \,\&\, x_8 \,\&\, x_9 \,\&\, x_{10} \,\&\, x_{11} \,\&\, x_{12} \,\&\, x_{13} \,\&\, x_{14} \,\&\, x_{15}$

  as

  $(x_1 \,\&\, x_2 \,\&\, x_3 \,\&\, x_4 \,\&\, x_5 \,\&\, x_6 \,\&\, x_7 \,\&\, x_8) \,\&\, (x_9 \,\&\, x_{10} \,\&\, x_{11} \,\&\, x_{12} \,\&\, x_{13} \,\&\, x_{14} \,\&\, x_{15}).$

These computations require 7 computers and 4 moments of time.

In general, parallel computations are needed to speed up computations. Without parallelism, we would need 14 moments of time to compute the desired value.